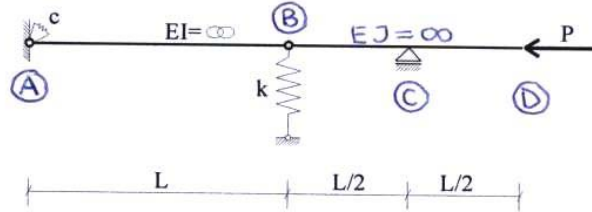
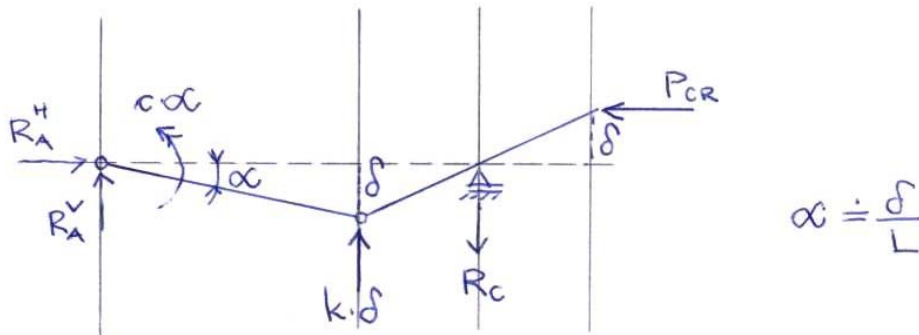


1. Odredite kritičnu silu sustava na slici:
 a. pristupom statičke ravnoteže
 b. energetske pristupom.



a) statička ravnoteža:



$$\sum M_A = 0 = P \cdot \delta - R_C \cdot 1,5L + k \cdot \delta \cdot L + c \cdot \alpha$$

$$R_C = \frac{P \cdot \delta + kL\delta + c \cdot \alpha}{1,5L}$$

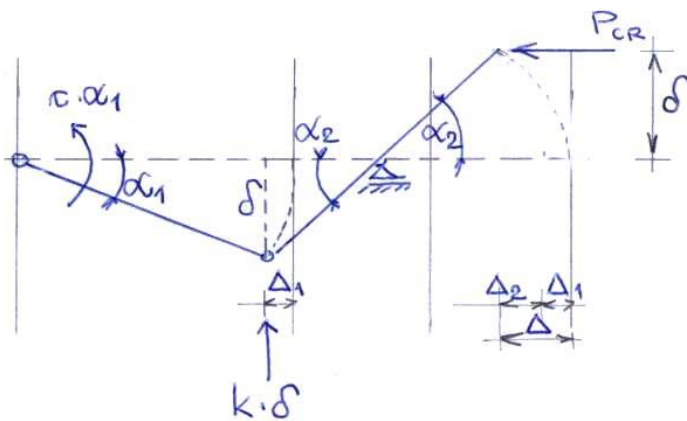
$$\sum M_B^D = 0 = P \cdot 2\delta - R_C \cdot 0,5L$$

$$2P\delta = \frac{P \cdot \delta + kL\delta + c \cdot \alpha}{1,5L} \cdot \frac{1}{2}$$

$$6P\delta = P \cdot \delta + kL\delta + c \frac{\delta}{L} \quad /: \delta$$

$$5P = kL + \frac{c}{L} \Rightarrow P_{ce} = \frac{kL^2 + c}{5L}$$

b) Energetski pristup



$$\alpha_1 \doteq \frac{\delta}{L}$$

$$\alpha_2 \doteq \frac{\delta}{\frac{L}{2}} = \frac{2\delta}{L}$$

$$\Delta_1 \doteq \frac{1}{2} L \cdot \alpha_1^2 = \frac{1}{2} L \cdot \frac{\delta^2}{L^2} = \frac{\delta^2}{2L}$$

$$\Delta_2 \doteq \frac{1}{2} L \cdot \alpha_2^2 = \frac{L}{2} \cdot \frac{4\delta^2}{L^2} = \frac{2\delta^2}{L}$$

- ukupna potencijalna energija: $\Pi = V + U$

$$V = -P \cdot \Delta = -P \cdot (\Delta_1 + \Delta_2) = -P \cdot \left(\frac{\delta^2}{2L} + \frac{2\delta^2}{L} \right) = -\frac{5P\delta^2}{2L}$$

$$U = \frac{1}{2} c \cdot \alpha_1^2 + \frac{1}{2} k \cdot \delta^2 = \frac{1}{2} c \cdot \frac{\delta^2}{L^2} + \frac{1}{2} k \delta^2$$

- za min. potencijala:

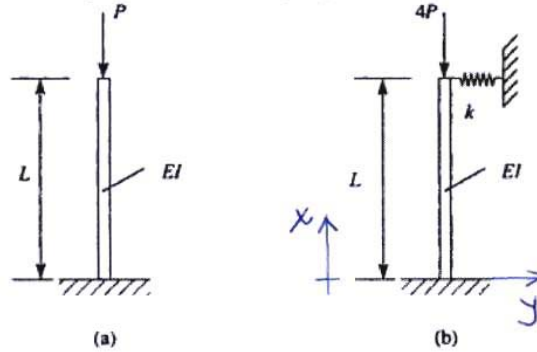
$$\delta \Pi = \frac{\partial \Pi}{\partial \delta} \delta \delta = 0 \Rightarrow \frac{\partial \Pi}{\partial \delta} = 0$$

$$-\frac{5P\delta}{L} + \frac{c \cdot \delta}{L^2} + k \cdot \delta = 0 \quad /: \delta / \cdot L^2$$

$$5PL = c + kL^2$$

$$P_{cr} = \frac{kL^2 + c}{5L}$$

2. Jednoliki stup duljine L i krutosti na savijanje EI upet je s donje a slobodan s gornje strane. Dimenzioniran je tako da može preuzeti kritičnu silu P (slika a). Naknadno je postavljen zahtjev za povećanom nosivosti što je izvedeno dodavanjem poprečne opruge na slobodnom kraju štapa (slika b). Odredite potrebnu krutost opruge, k , tako da kritična sila bude $4P$.



Rubni uvjeti:

$$\begin{aligned} x=0: y &= 0 & (1) \\ x=0: y' &= 0 & (2) \\ x=L: M &= 0 & (3) \\ x=L: T &= k \cdot y & (4) \end{aligned}$$

$$(1): y(x=0) = B + D = 0 \Rightarrow D = -B$$

$$(2): y'(x=0) = \alpha A + C = 0 \Rightarrow C = -\alpha A$$

$$(3): M(x=L) = -EJ y''(x=L) = -EJ \cdot (-\alpha^2 A \sin \alpha L - \alpha^2 B \cos \alpha L) = 0$$

$$A \sin \alpha L + B \cos \alpha L = 0$$

$$(4): T(x=L) = -EJ [y'''(x=L) + \alpha^2 y'(x=L)] = -k \cdot y(x=L)$$

$$EJ(-\alpha^3 A \cos \alpha L + \alpha^3 B \sin \alpha L + \alpha^3 A \cos \alpha L - \alpha^3 B \sin \alpha L + \alpha^2 C)$$

$$= k \cdot (A \sin \alpha L + B \cos \alpha L + CL + D)$$

$$A \sin \alpha L + B \cos \alpha L = 0$$

$$kA \sin \alpha L + kB \cos \alpha L - k\alpha AL - kB + EJ\alpha^3 A = 0$$

$$\begin{bmatrix} \sin \alpha L & \cos \alpha L \\ k \sin \alpha L - k\alpha L + EJ\alpha^3 & k \cos \alpha L - k \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

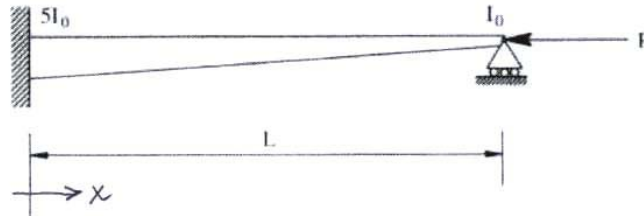
$$k \sin \alpha L \cos \alpha L - k \sin \alpha L - k \sin \alpha L \cos \alpha L + k\alpha L \cos \alpha L - EJ\alpha^3 \cos \alpha L = 0$$

/: $\cos \alpha L$

$$k(\alpha L - \tan \alpha L) = EJ\alpha^3, \quad \alpha^2 = \frac{4P}{EJ}$$

$$k = \frac{4P\alpha}{\alpha L - \tan \alpha L}$$

3. Odredite kritičnu silu konzole duljine L promjenjive krutosti pomoću Rayleigh-evog kvocijenta.



- FUNKCIJA PROMJENE MOMENTA TRUOSTI DUŽ OSI ŠTAPA:

$$I(x) = I_1 + \frac{x}{L}(I_0 - I_1)$$

za $I_1 = 5I_0$:
$$I(x) = I_0 \left(5 - \frac{4x}{L} \right)$$

- PRIJEDLOG FUNKCIJE PROGIBNE LINIJE, $y(x)$
 geometrijski rubni uvjeti konstrukciji:

$$y(0) = y'(0) = 0 \quad \text{i} \quad y(L) = 0 \quad \Rightarrow \quad n = 3$$

n (stupanj polinoma) = $n + 1 = 4$

- dodatni fizikalni rubni uvjet: $y''(L) = 0$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$y(x=0) = 0 \Rightarrow a_0 = 0$$

$$y'(x=0) = 0 \Rightarrow a_1 = 0$$

$$y(x=L) = a_2 L^2 + a_3 L^3 + a_4 L^4 = 0$$

$$y''(x=L) = 2a_2 + 6a_3 L + 12a_4 L^2 = 0$$

pretpostavimo da je $a_4 L^4 = a$ (nova konstanta)

tada iz posljednje dvije jednačbe slijedi:

$$a_2 = \frac{3}{2} \cdot \frac{a}{L^2} \quad \text{i} \quad a_3 = -\frac{5}{2} \cdot \frac{a}{L^3}$$

- KONAČNO JE :

$$y(x) = a \left[\frac{3}{2} \left(\frac{x}{L} \right)^2 - \frac{5}{2} \left(\frac{x}{L} \right)^3 + \left(\frac{x}{L} \right)^4 \right]$$

RAYLEIGH :
$$P_{cr} = \frac{\int_0^L EJ(y''(x))^2 dx}{\int_0^L (y'(x))^2 dx}$$

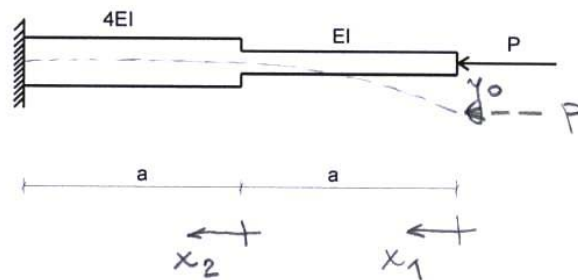
$$y'(x) = a \left[\frac{3x}{L^2} - \frac{15x^2}{2L^3} + \frac{4x^3}{L^4} \right], \quad y''(x) = a \left[\frac{3}{L^2} - \frac{15x}{L^3} + \frac{12x^2}{L^4} \right]$$

$$\begin{aligned} \int_0^L EJ(x) \cdot (y''(x))^2 dx &= \int_0^L EI_0 \left(5 - \frac{4x}{L}\right) \left[a \left(\frac{3}{L^2} - \frac{15x}{L^3} + \frac{12x^2}{L^4} \right) \right]^2 dx = \\ &= \frac{6EJ_0 a^2}{L^3} \end{aligned}$$

$$\int_0^L (y'(x))^2 dx = \int_0^L \left[a \left(\frac{3x}{L^2} - \frac{15x^2}{2L^3} + \frac{4x^3}{L^4} \right) \right]^2 dx = \frac{0,086 a^2}{L}$$

$$\begin{aligned} P_{cr} &= \frac{\frac{6EJ_0 a^2}{L^3}}{\frac{0,086 a^2}{L}} = \frac{69,8 EJ_0}{L^2} = \frac{7,1 \pi^2 EJ_0}{L^2} \\ &= \frac{\pi^2 EJ_0}{(0,376L)^2} \end{aligned}$$

4. Odredite kritičnu silu konzole duljine $2a$ promjenjive krutosti, metodom početnih parametara.



$$\alpha_1^2 = \frac{P}{EJ} \equiv \alpha^2 ; \quad \alpha_2^2 = \frac{P}{4EJ} = 0,5\alpha$$

$$0 \leq x_1 \leq a$$

$$y(x_1=a) = y_a = y_0 + y_0' \frac{\sin \alpha_1 a}{\alpha_1}$$

$$y'(x_1=a) = y_a' = y_0' \cos \alpha_1 a$$

$$M(x_1=a) = M_a = \alpha_1 EJ y_0' \sin \alpha_1 a$$

$$T(x_1=a) = T_0 = \emptyset$$

$$0 \leq x_2 \leq a$$

$$y(x_2=a) = y_a + y_a' \frac{\sin \alpha_2 a}{\alpha_2} - M_a \frac{1 - \cos \alpha_2 a}{\alpha_2^2 \cdot 4EJ} = \emptyset$$

$$y'(x_2=a) = y_a' \cos \alpha_2 a - M_a \frac{\sin \alpha_2 a}{\alpha_2 \cdot 4EJ} = \emptyset$$

$$y(x_2=a) = y_0 + y_0' \frac{\sin \alpha a}{\alpha} + y_0' \cos \alpha a \cdot \frac{\sin 0,5\alpha a}{0,5\alpha} - EJ\alpha y_0' \sin \alpha a \cdot \frac{1 - \cos 0,5\alpha a}{4EJ \cdot (0,5\alpha)^2} = \emptyset$$

$$y'(x_2=a) = y_0' \cos \alpha a \cdot \cos 0,5\alpha a - EJ\alpha y_0' \sin \alpha a \frac{\sin 0,5\alpha a}{0,5\alpha \cdot 4EJ} = \emptyset$$

$$y_0 + y_0' \frac{\sin \alpha a}{\alpha} + y_0'' \cos \alpha a \cdot \frac{\sin 0,5 \alpha a}{0,5 \alpha} -$$
$$- y_0''' \sin \alpha a \cdot \frac{1 - \cos 0,5 \alpha a}{\alpha} = 0 \quad (1)$$

$$y_0'' \cos \alpha a \cdot \cos 0,5 \alpha a - y_0''' \sin \alpha a \cdot \frac{\sin 0,5 \alpha a}{2} = 0 \quad (2)$$

Iz jednačbe (2) slijedi:

$$y_0'' (2 \cos \alpha a \cdot \cos 0,5 \alpha a - \sin \alpha a \cdot \sin 0,5 \alpha a) = 0$$

$$2 \cos \alpha a \cdot \cos 0,5 \alpha a - \sin \alpha a \cdot \sin 0,5 \alpha a = 0$$

supstitucija: $u = \alpha a$

$$f(u) = 2 \cos u \cdot \cos 0,5 u - \sin u \cdot \sin 0,5 u = 0$$

Metodom pokušaja: $u = 1,231$

$$\alpha = \frac{1,231}{a} = \sqrt{\frac{P_{cr}}{EJ}} \quad /^2$$

$$P_{cr} = \frac{1,52 EJ}{a^2}$$