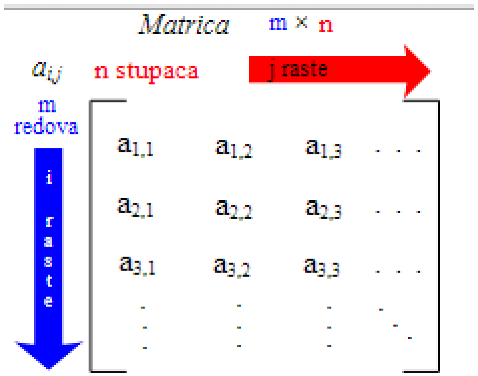
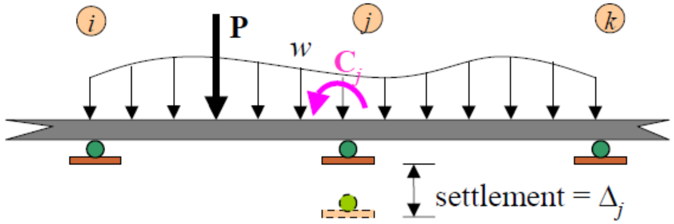
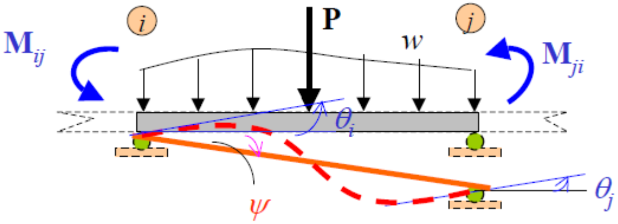
# MATRIČNA PRISTUP METODI POMAKA



# METODA POMAKA

Zadan st.neodređen, translatorno nepomični sustav.
Nepoznanica: φj





**Metoda pomaka-rješenje st.n.sustava traži na zamjenskom-osnovnom sustavu**

dodaje veze zad. sustavu na mjestu nepoznatih pomaka (koje sprječavaju pomak), a onda rastavlja zadani statički sustav na pojedine elemente.

1. se postavlja veza P-u (M-φ,ψ) za svaki element u lks.

   Koristi se superpozicija:

   a) stanje upetosti -M +

   b) stanje slobodnih pomaka- m

2. se postavljaju jednadžbe iz kojih tražimo nepoznate veličine

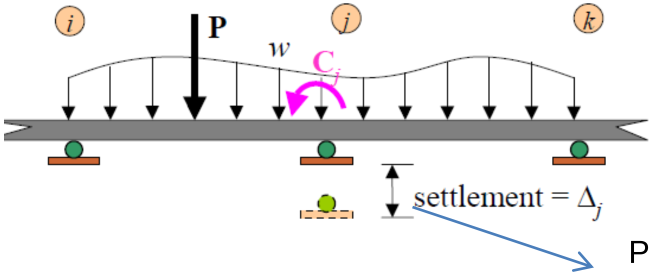
   (jednadžbe ravnoteže slobodnih čvorova i jedn.rada za v.p.)

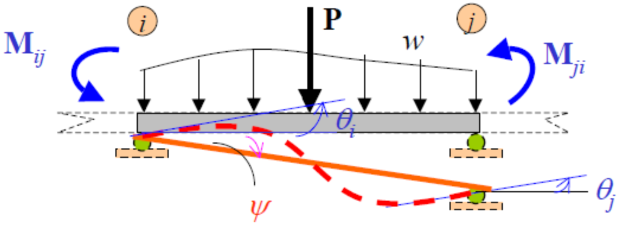
# METODA POMAKA

## a)STANJE UPETOSTI

M -od vanjskog opterećenja, na pojedinim elementima.

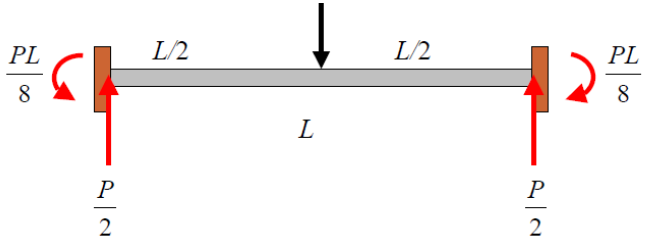
To je poznata veličina.

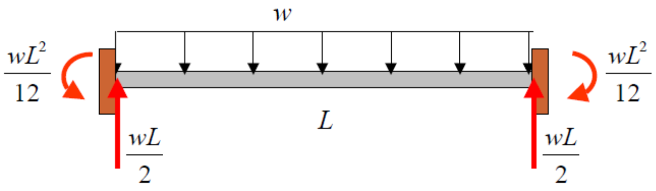




Pomak zadan kao opterećenje-daje M

M=6*k *ψ
Moment upetosti se računa preko izraza za st. sl. pom.

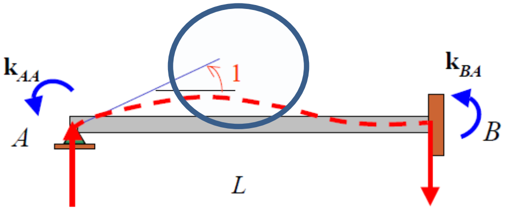




# METODA POMAKA

## b) STANJE SLOBODNIH POMAKA

m –od prisilnih pomaka na mjestu sprječenih pomaka(poništavaju pridržanja), na pojedinim elementima. Ne znamo veličinu istih.



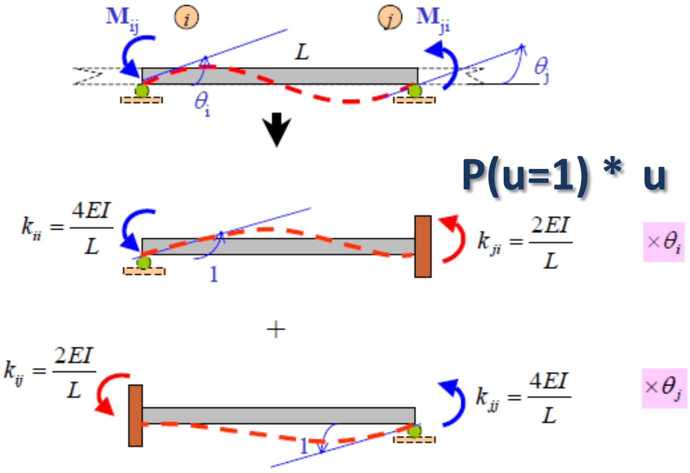
$$k_{AA} = \frac{4EI}{L}$$

$$k_{BA} = \frac{2EI}{L}$$

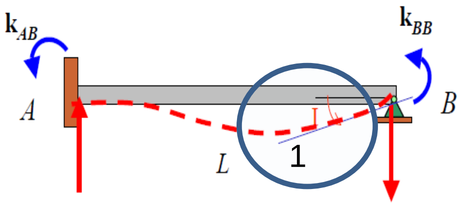
Koristimo superpoziciju:

**P(u) = P(u=1) * u**

$$\text{sila za pomak u} = \underbrace{\text{sila za pomak u=1}}_{\text{krutost}} \;*\; \mathbf{u}$$



**P(u=1) * u**



$$k_{BB} = \frac{4EI}{L}$$

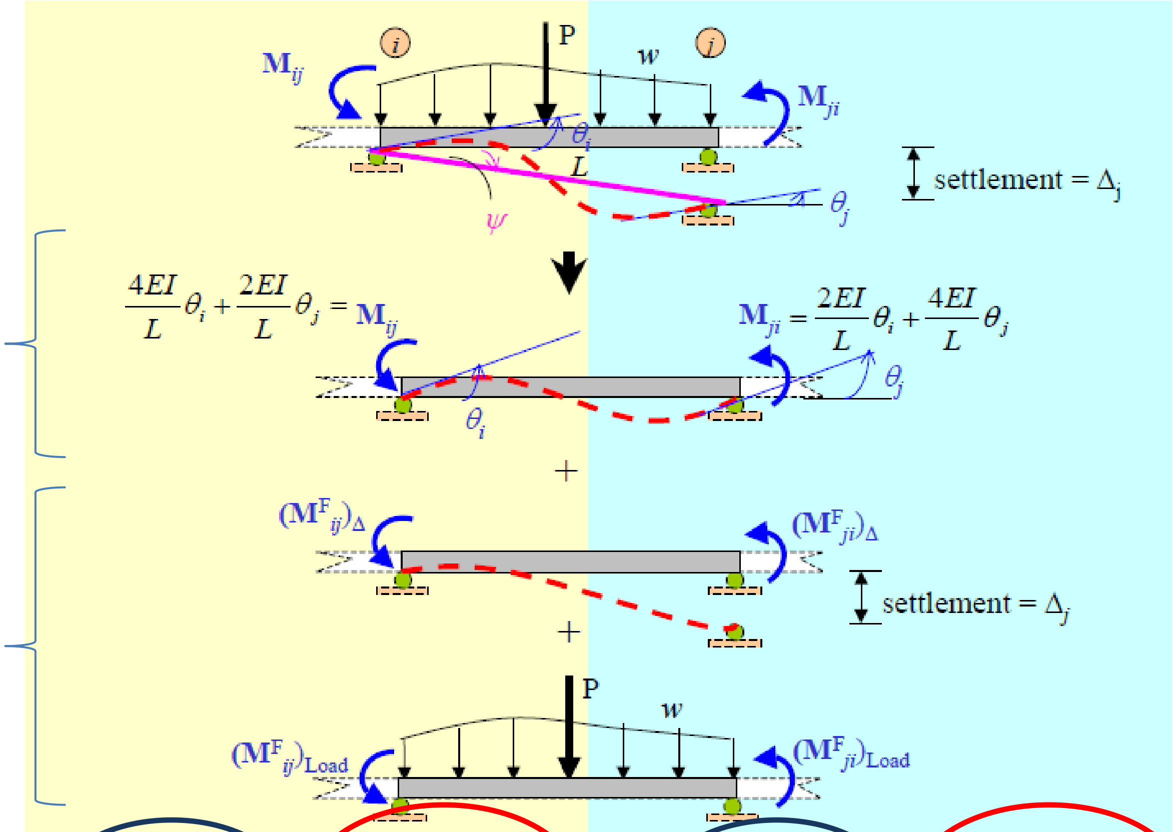
$$k_{AB} = \frac{2EI}{L}$$

krutost =sila(m) za pomak u(φ)=1

# METODA POMAKA

**STANJE SLOB. POMAKA**

**STANJE UPETOSTI**



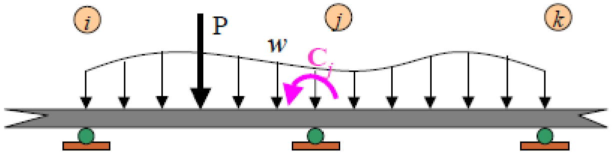
$$\frac{4EI}{L}\theta_i + \frac{2EI}{L}\theta_j = M_{ij}$$

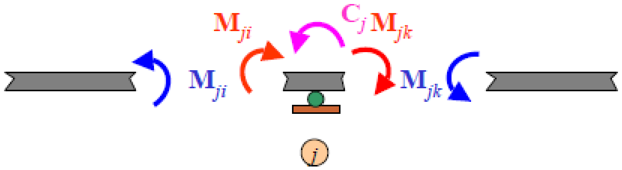
$$M_{ji} = \frac{2EI}{L}\theta_i + \frac{4EI}{L}\theta_j$$

$$\underbrace{M_{ij} = (\frac{4EI}{L})\theta_i + (\frac{2EI}{L})\theta_j + (M^F{}_{ij})_\Delta + (M^F{}_{ij})_{Load} \; M_{ji} = (\frac{2EI}{L})\theta_i + (\frac{4EI}{L})\theta_j + (M^F{}_{ji})_\Delta + (M^F{}_{ji})_{Load}}_{\text{P-u}}$$

# METODA POMAKA

## 2. JEDNADŽBE METODE POMAKA





$$+\circlearrowright\ \Sigma M_j = 0: \quad -M_{ji} - M_{jk} + C_j = 0$$

Jednadžbe ravnoteže slobodnih čvorova

Momente sa kraja štapa prebacujemo u čvor.

$$M_{ij} = (\frac{4EI}{L})\theta_i + (\frac{2EI}{L})\theta_j + (M^F{}_{ij})$$

$$M_{ji} = (\frac{2EI}{L})\theta_i + (\frac{4EI}{L})\theta_j + (M^F{}_{ji})$$

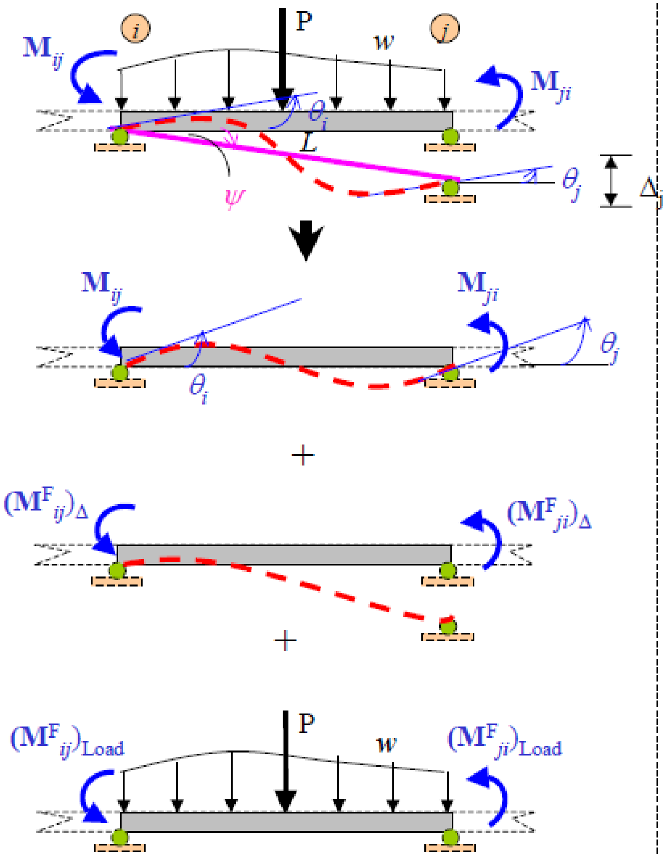
$$\begin{bmatrix} M_{ij} \\ M_{ji} \end{bmatrix} = \begin{bmatrix} (4EI/L) & (2EI/L) \\ (2EI/L) & (4EI/L) \end{bmatrix} \begin{bmatrix} \theta_{iI} \\ \theta_j \end{bmatrix} + \begin{bmatrix} M_{ij}{}^F \\ M_{ji}{}^F \end{bmatrix}$$

Matrična forma jedn.sila na kraju štapa

$$[k] = \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix}$$

Krutost-veza između sila i deformacija

# NEPOZNANICE-POMACI NA KRAJU ELEMENATA



$$[M]=[K][\theta]+[FEM]$$

$$([M]-[FEM])=[K][\theta]$$

$$[\theta]=[K]^{-1}[M]-[FEM]$$

Stiffness matrix

Fixed-end moment matrix

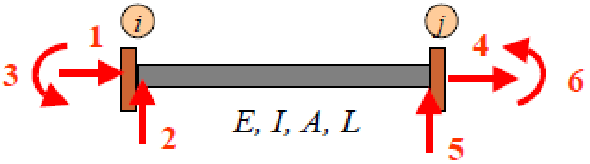
$$\boxed{[D] = [K]^{-1}([Q] - [FEM])}$$

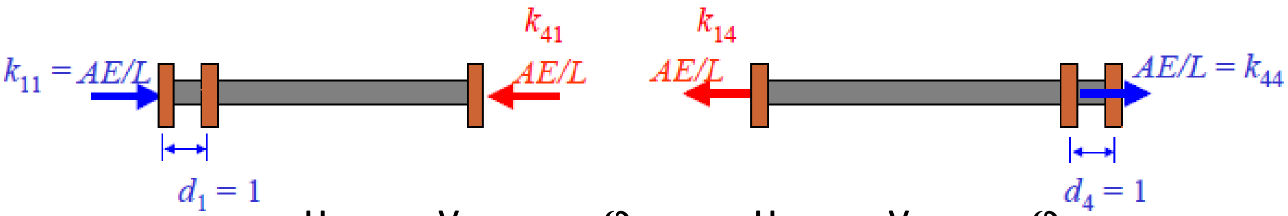
Displacement matrix

Force matrix

# MATRICA KRUTOSTI GREDE

AKSIJALNA KRUTOST krutost =sila(N) za pomak u=1





$$[k] = \begin{array}{cc|cccccc} & & u & v & \varphi & u & v & \varphi \\ & & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline N & 1 & AE/L & & & -AE/L & & \\ T & 2 & 0 & & & 0 & & \\ M & 3 & 0 & & & 0 & & \\ N & 4 & -AE/L & & & AE/L & & \\ T & 5 & 0 & & & 0 & & \\ M & 6 & 0 & & & 0 & & \end{array}$$

**K ij**

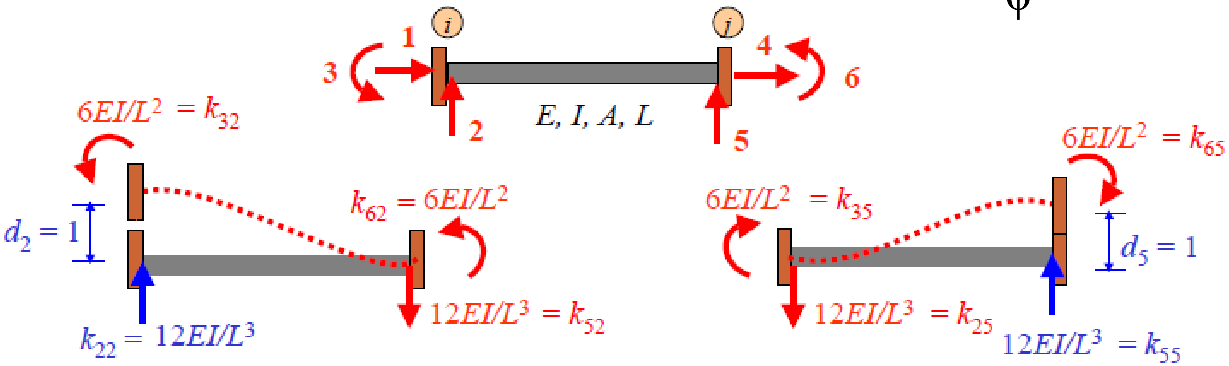
**i = mjesto sile**

**j = mjesto pomaka**

# MATRICA KRUTOSTI GREDE

POSMIČNA KRUTOST krutost =sila(T, M) za pomak v=1

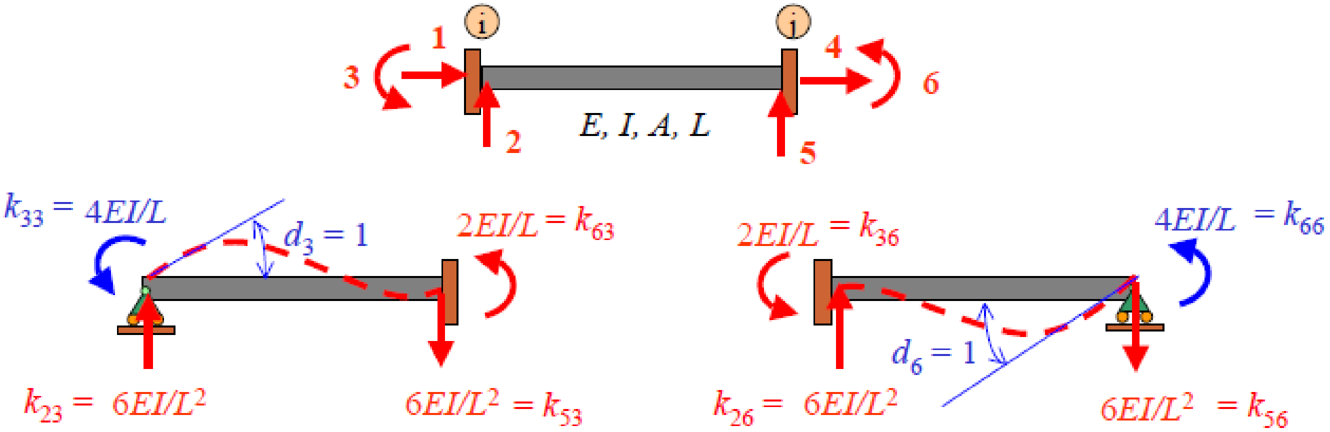
$\psi$



$$[k] = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & AE/L & 0 & & -AE/L & 0 & \\ 2 & 0 & 12EI/L^3 & & 0 & -12EI/L^3 & \\ 3 & 0 & 6EI/L^2 & & 0 & -6EI/L^2 & \\ 4 & -AE/L & 0 & & AE/L & 0 & \\ 5 & 0 & -12EI/L^3 & & 0 & 12EI/L^3 & \\ 6 & 0 & 6EI/L^2 & & 0 & -6EI/L^2 & \end{array}$$

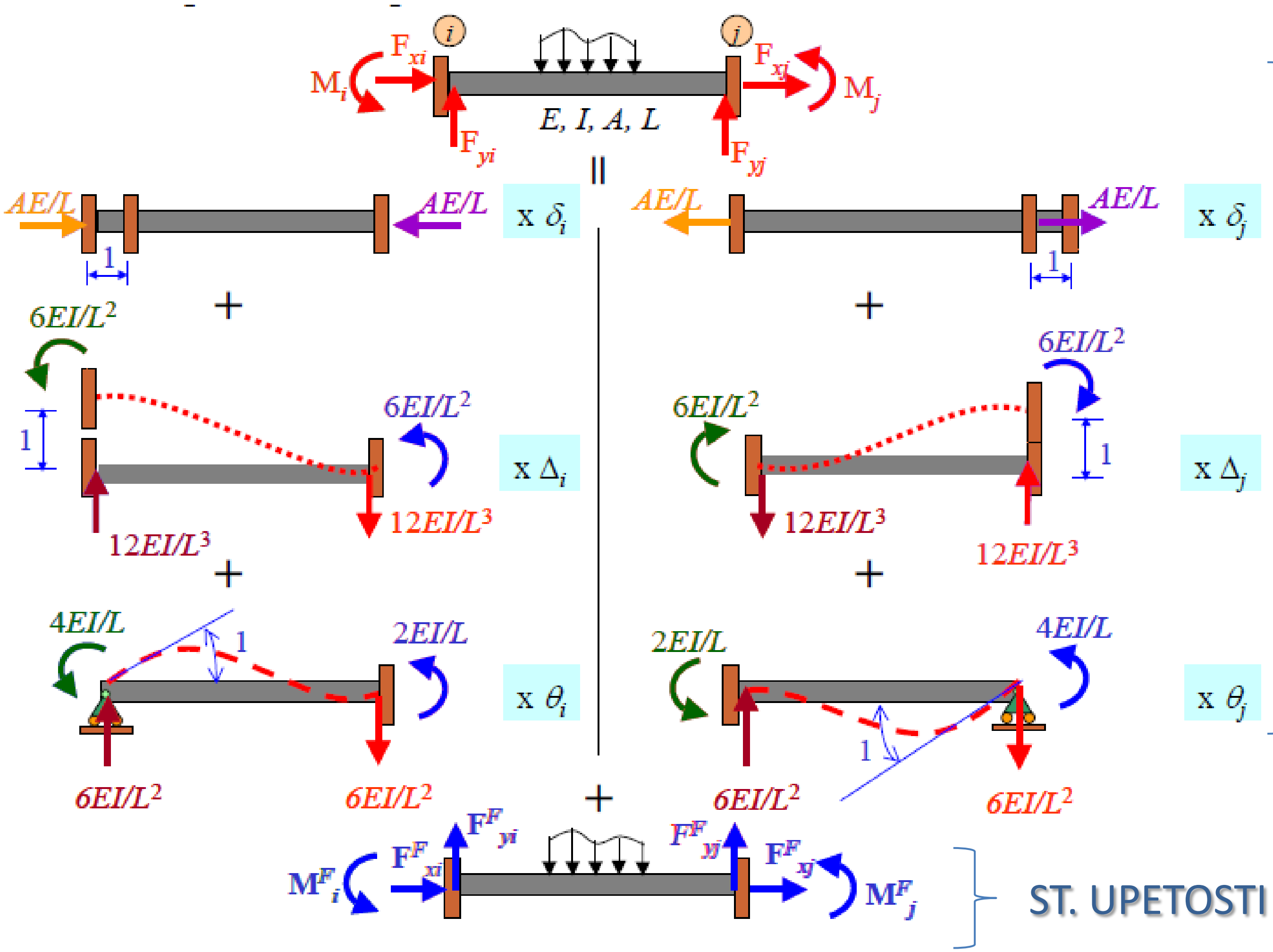
# MATRICA KRUTOSTI GREDE-ŠTAPA

FLEKSIJSKA KRUTOST krutost =sila(T, M) za pomak φ=1



$$[k] = \begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 2 & 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 3 & 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ 4 & -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 5 & 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 6 & 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{array}$$

# MATRICA KRUTOSTI GREDE
# JEDNADŽBE SILA NA KRAJU ŠTAPOVA



S
T

S
L
O
B

P
O
M
A
K
A

ST. UPETOSTI

# SILE NA KRAJU GREDA
# P-u

$$
\begin{array}{lllllllll}
F_{xi} = & (AE/L)\delta_i + & (0)\Delta_i & (0)\theta_i + & (-AE/L)\delta_j + & (0)\Delta_j + & (0)\theta_j + & F_{xi}^F \\
F_{yi} = & (0)\delta_i + & (12EI/L^3)\Delta_i & (6EI/L^2)\theta_i & (0)\delta_j & (-12EI/L^3)\Delta_j & (6EI/L^2)\theta_j & F_{yi}^F \\
M_{xi} = & (0)\delta_i & (6EI/L^2)\Delta_i & (4EI/L)\theta_i & (0)\delta_j & (-6EI/L^2)\Delta_j & (2EI/L)\theta_j & M_i^F \\
F_{xj} = & (-AE/L)\delta_i & (0)\Delta_i & (0)\theta_i & (AE/L)\delta_j & (0)\Delta_j & (0)\theta_j & F_{xi}^F \\
F_{yj} = & (0)\delta_i & (-12EI/L^3)\Delta_i & (-6EI/L^2)\theta_i & (0)\delta_j & (0)\Delta_j & (-6EI/L^2)\theta_j & F_{yj}^F \\
M_j = & (0)\delta_i & (6EI/L^2)\Delta_i & (2EI/L)\theta_i & (0)\delta_j & (-6EI/L^2)\Delta_j & (4EI/L)\theta_j & M_j^F
\end{array}
$$

$$
\begin{bmatrix} F_{xi} \\ F_{yj} \\ M_i \\ F_{xj} \\ F_{yj} \\ M_j \end{bmatrix}
=
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\begin{bmatrix} \delta_i \\ \Delta_i \\ \theta_i \\ \delta_j \\ \Delta_j \\ \theta_j \end{bmatrix}
+
\begin{bmatrix} F_{xi}^F \\ F_{yi}^F \\ M_i^F \\ F_{xj}^F \\ F_{yi}^F \\ M_j^F \end{bmatrix}
$$

$$\Downarrow$$

Stiffness matrix

Fixed-end force matrix

$$\boxed{[q] = [k][d] + [q^{\mathrm{F}}]}$$

End-force matrix

Displacement matrix

# MATRICA KRUTOSTI GREDE

- **6x6 Stiffness Matrix**

$$[k]_{6\times 6} = \begin{array}{c|cccccc} & \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \\ \hline N_i & AE/L & 0 & 0 & -AE/L & 0 & 0 \\ V_i & 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ M_i & 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ N_j & -AE/L & 0 & 0 & AE/L & 0 & 0 \\ V_j & 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ M_j & 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{array}$$

- **4x4 Stiffness Matrix**

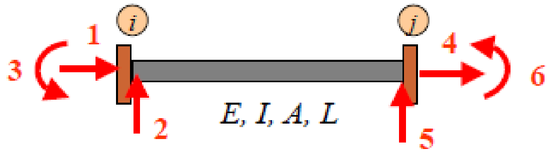
$$[k]_{4\times 4} = \begin{array}{c|cccc} & \Delta_i & \theta_i & \Delta_j & \theta_j \\ \hline V_i & 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ M_i & 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ V_j & -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ M_j & 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{array}$$

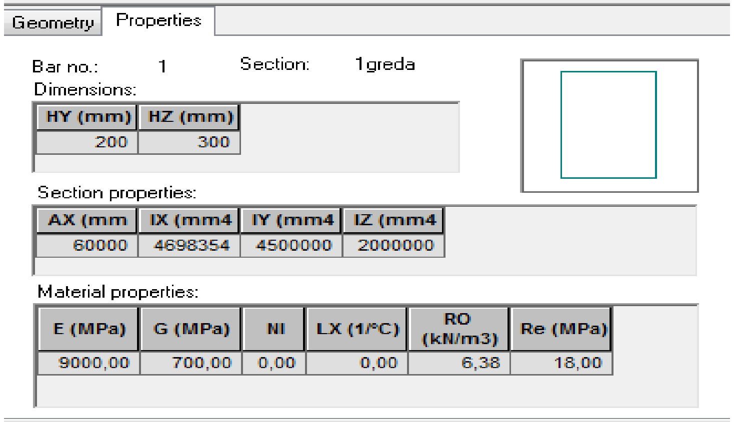
- **2x2 Stiffness Matrix**

$$[k]_{2\times 2} = \begin{array}{c|cc} & \theta_i & \theta_j \\ \hline M_i & 4EI/L & 2EI/L \\ M_j & 2EI/L & 4EI/L \end{array}$$

# ZADATAK

Odrediti računalom matricu krutosti dvostrano upete grede raspona L=8 m.





Bar no.: 1 Section: 1greda

Dimensions:

| HY (mm) | HZ (mm) |
|---|---|
| 200 | 300 |

Section properties:

| AX (mm | IX (mm4 | IY (mm4 | IZ (mm4 |
|---|---|---|---|
| 60000 | 4698354 | 4500000 | 2000000 |

Material properties:

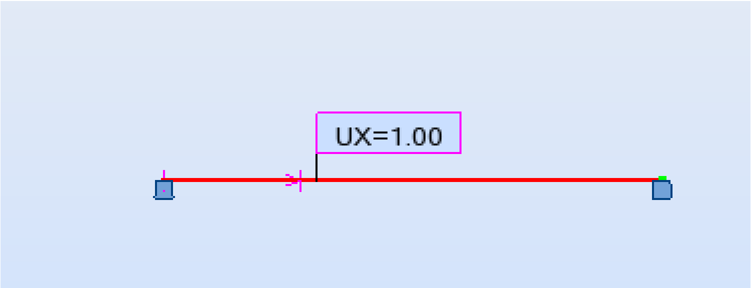
| E (MPa) | G (MPa) | NI | LX (1/°C) | RO (kN/m3) | Re (MPa) |
|---|---|---|---|---|---|
| 9000,00 | 700,00 | 0,00 | 0,00 | 6,38 | 18,00 |

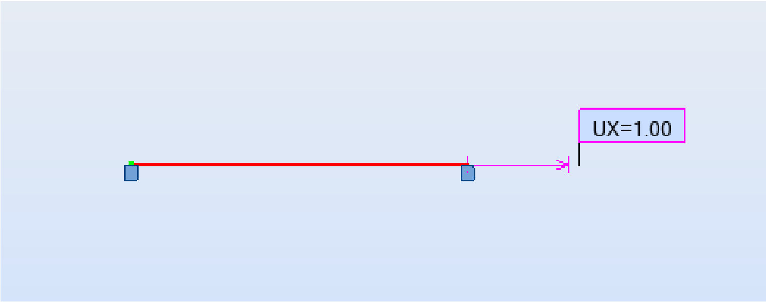
- **6x6 Stiffness Matrix**

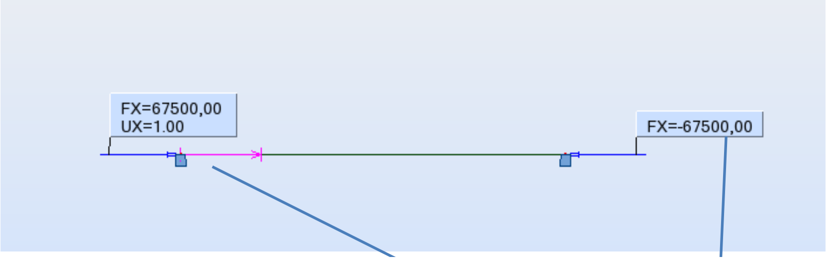
$$[k]_{6\times 6} = \begin{array}{cc} & \begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\ \begin{array}{c} N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array} & \begin{bmatrix} \quad & \quad & \quad & \quad & \quad & \quad \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

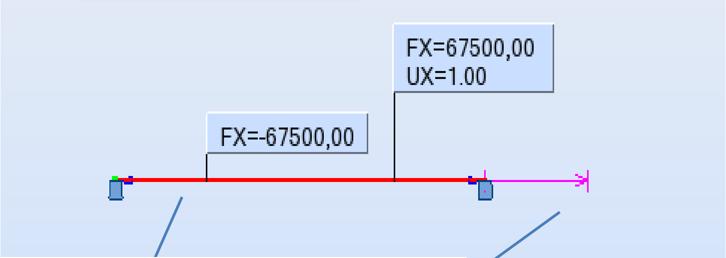
# ZADATAK

AKSIJALNA KRUTOST      krutost =sila(N) za pomak u=1





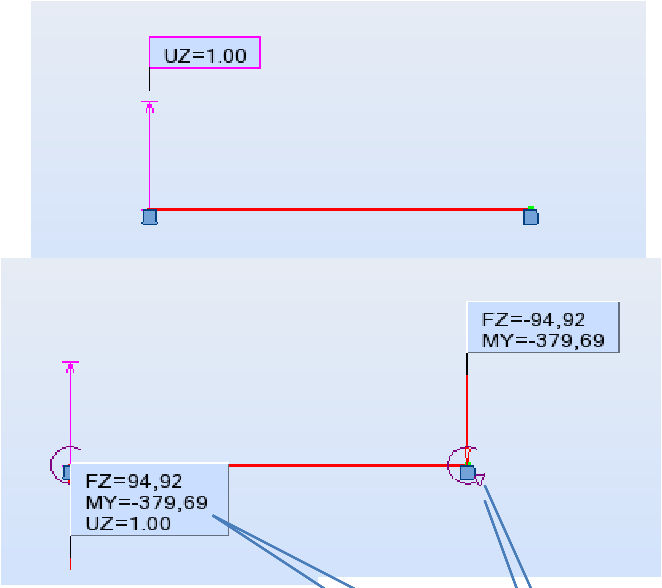


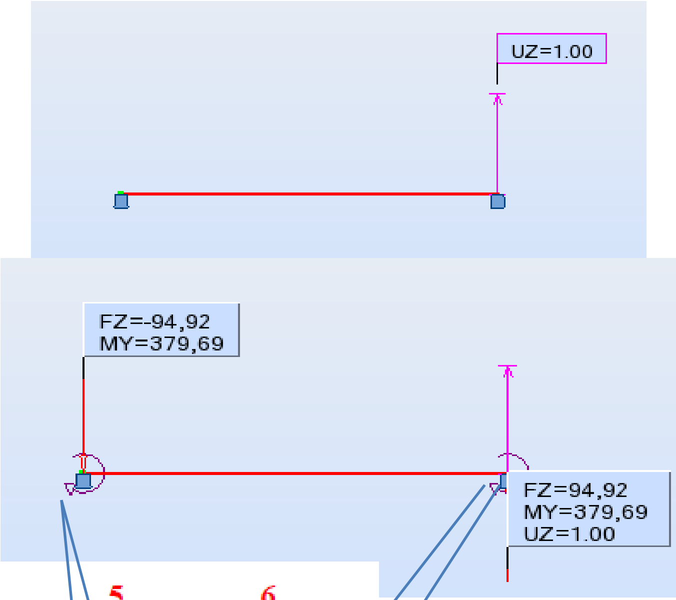


$$
[k]\quad
\begin{array}{ll|cccccc|}
 & & \delta & \Delta & \theta & \delta & \Delta & \theta \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
N_i & 1 & AE/L & & & -AE/L & & \\
V_i & 2 & 0 & & & 0 & & \\
M_i & 3 & 0 & & & 0 & & \\
N_j & 4 & -AE/L & & & AE/L & & \\
V_j & 5 & 0 & & & 0 & & \\
M_j & 6 & 0 & & & 0 & & \\
\end{array}
$$

# ZADATAK

POSMIČNA KRUTOST krutost =sila(T, M) za pomak v=1

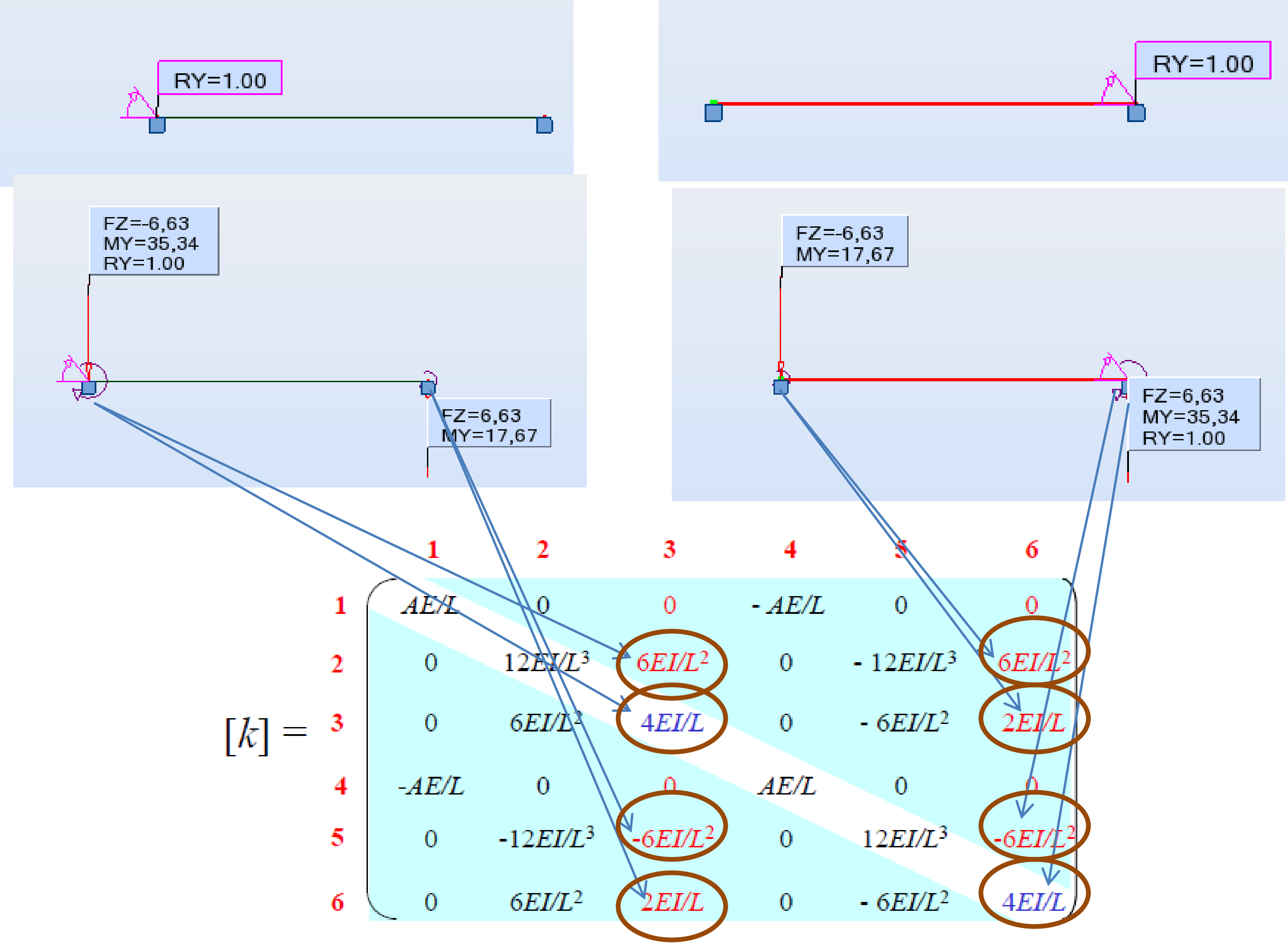




$$[k] = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & AE/L & 0 & & -AE/L & 0 & \\ 2 & 0 & 12EI/L^3 & & 0 & -12EI/L^3 & \\ 3 & 0 & 6EI/L^2 & & 0 & -6EI/L^2 & \\ 4 & -AE/L & 0 & & AE/L & 0 & \\ 5 & 0 & -12EI/L^3 & & 0 & 12EI/L^3 & \\ 6 & 0 & 6EI/L^2 & & 0 & -6EI/L^2 & \end{array}$$

# ZADATAK

FLEKSIJSKA KRUTOST krutost =sila(T, M) za pomak φ=1



$$[k] = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 2 & 0 & 12EI/L^3 & \boxed{6EI/L^2} & 0 & -12EI/L^3 & \boxed{6EI/L^2} \\ 3 & 0 & 6EI/L^2 & \boxed{4EI/L} & 0 & -6EI/L^2 & \boxed{2EI/L} \\ 4 & -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 5 & 0 & -12EI/L^3 & \boxed{-6EI/L^2} & 0 & 12EI/L^3 & \boxed{-6EI/L^2} \\ 6 & 0 & 6EI/L^2 & \boxed{2EI/L} & 0 & -6EI/L^2 & \boxed{4EI/L} \end{array}$$

# ZADATAK

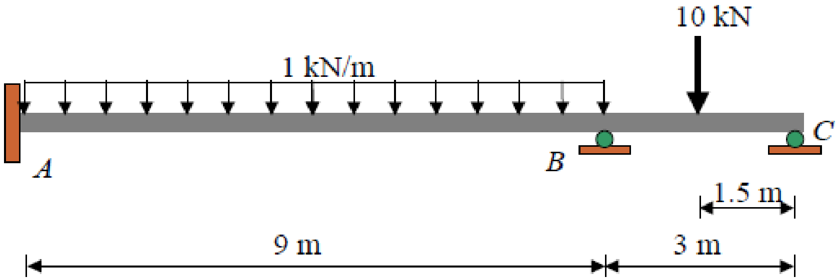
1. Što će se promijeniti ako promjenimo dimenzije poprečnog presjeka elementa grede ili materijal grede u ovom zadatku ?

2. Što bi se promijenilo da je statički određen nosač ?

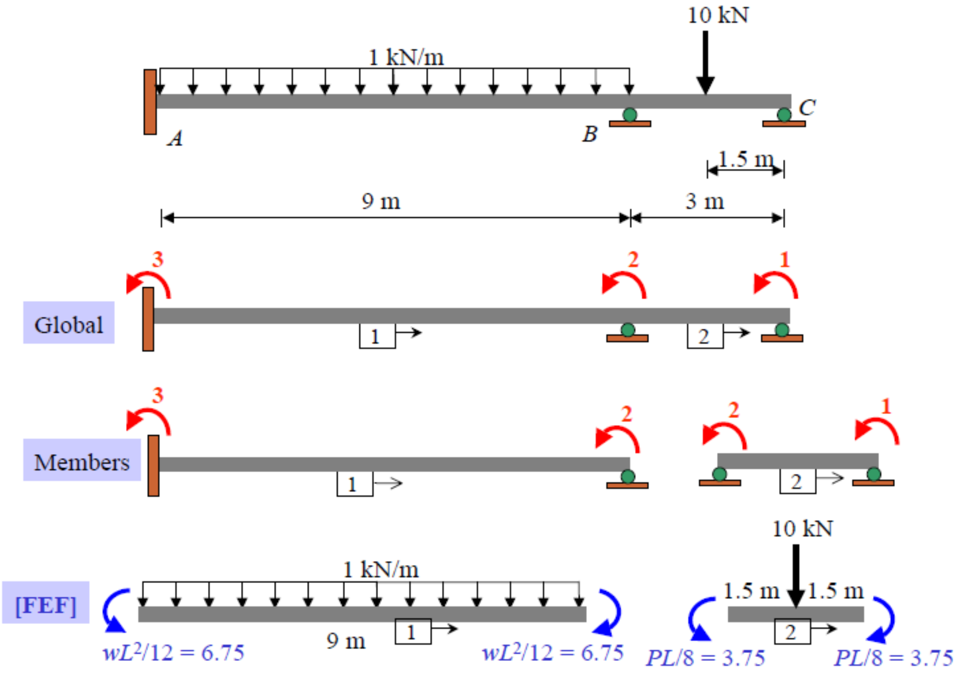
# PRIMJER MATRIČNOG PRISTUPA MET.POM.

Za prikazanu gredu:

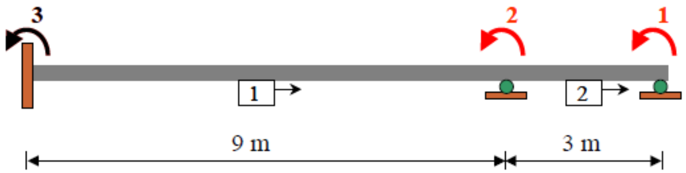
a) Odrediti pomak i rotaciju u B
b) Odrediti sve reakcije oslonaca
c) Nacrtati dijagrame M i T sila



# MATRIČNI PRISTUP MET.POM.



# MATRIČNI PRISTUP MET.POM.



**Stiffness Matrix:**

$$[k]_{2\times2} = \begin{array}{ccc} & \theta_i & \theta_j \\ M_i & 4EI/L & 2EI/L \\ M_j & 2EI/L & 4EI/L \end{array}$$

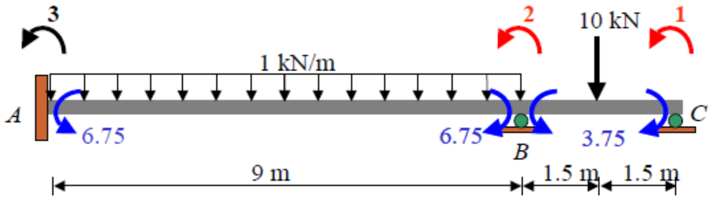
$$[k]_1 = EI \begin{array}{ccc} 3 & 2 & \\ 4/9 & 2/9 & 3 \\ 2/9 & \boxed{4/9} & 2 \end{array} \qquad [k]_2 = EI \begin{array}{ccc} 2 & 1 & \\ \boxed{4/3} & 2/3 & 2 \\ 2/3 & 4/3 & 1 \end{array}$$

$$[K] = EI \begin{array}{ccc} 2 & 1 & \\ \boxed{(4/9)+(4/3)} & 2/3 & 2 \\ 2/3 & 4/3 & 1 \end{array}$$

⇒

u čvoru 2
nepoznati kut –
u njemu jedn.
ravn.

# MATRIČNI PRISTUP MET.POM.



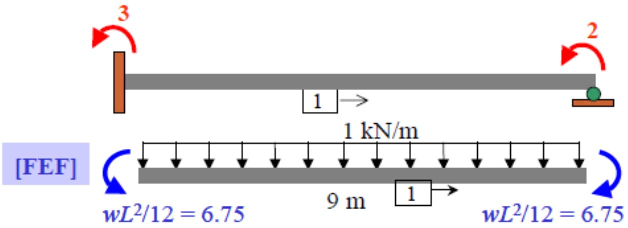
**Equilibrium equations:** $M_{CB} = 0$

$M_{BA} + M_{BC} = 0$

**Global Equilibrium:** $[Q] = [K][D] + [Q^F]$

$$\begin{array}{c} 2 \\ 1 \end{array}\begin{bmatrix} M_{BA}+M_{BC}=0 \\ M_{CB}=0 \end{bmatrix} = EI \begin{array}{c} \\ 2 \\ 1 \end{array}\begin{bmatrix} \overset{2}{\boxed{(4/9)+(4/3)}} & \overset{1}{2/3} \\ 2/3 & 4/3 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} + \begin{bmatrix} -6.75+3.75=-3 \\ -3.75 \end{bmatrix}$$

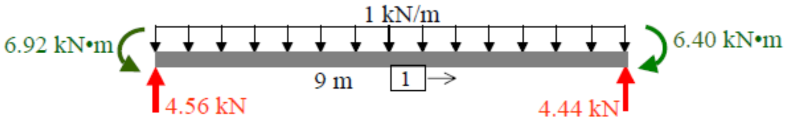
$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 0.779/EI \\ 2.423/EI \end{bmatrix}$$

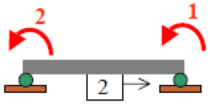


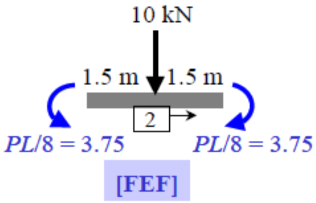
Substitute $\theta_B$ and $\theta_C$ in the member matrix,

**Member** $\boxed{1}$ : $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{array}{c} 3 \\ 2 \end{array}\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = EI\begin{array}{c} \begin{array}{cc} 3 & \quad 2 \end{array} \\ \begin{bmatrix} 4/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix} \end{array}\begin{bmatrix} \overset{0}{\cancel{\theta_A}} \\ \theta_B = 0.779/EI \end{bmatrix} + \begin{bmatrix} 6.75 \\ -6.75 \end{bmatrix} = \begin{bmatrix} 6.92 \\ -6.40 \end{bmatrix}$$



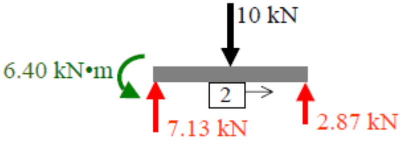


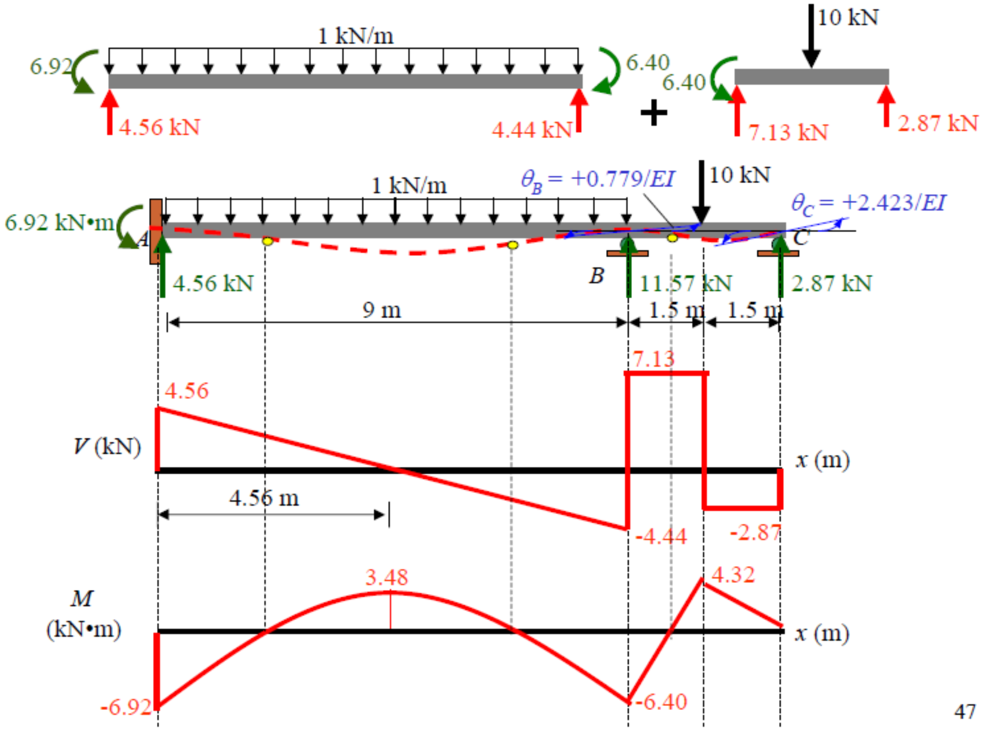


Substitute $\theta_B$ and $\theta_C$ in the member matrix,

**Member** $\boxed{2}$ : $[q]_2 = [k]_2[d]_2 + [q^F]_2$

$$\begin{array}{c} 2 \\ 1 \end{array}\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = EI \begin{array}{c} \begin{array}{cc} 2 & 1 \end{array} \\ \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \end{array} \begin{bmatrix} \theta_B = 0.779/EI \\ \theta_C = 2.423/EI \end{bmatrix} + \begin{bmatrix} 3.75 \\ -3.75 \end{bmatrix} = \begin{bmatrix} 6.40 \\ 0 \end{bmatrix}$$




1 kN/m
6.92
6.40
4.56 kN
4.44 kN
+
10 kN
6.40
7.13 kN
2.87 kN
1 kN/m
$\theta_B = +0.779/EI$
10 kN
$\theta_C = +2.423/EI$
6.92 kN•m
A
B
C
4.56 kN
11.57 kN
2.87 kN
9 m
1.5 m
1.5 m
7.13
4.56
V (kN)
x (m)
4.56 m
-4.44
-2.87
3.48
4.32
M
(kN•m)
x (m)
-6.92
-6.40