Građevna statika 2 (21093) | Metoda proračuna 3/3
Akademska godina 2020./2021.

Iterativne metode | Cross & Csonka-Werner

Uvod, objašnjenje postupka, osnovni proračun, riješeni primjeri

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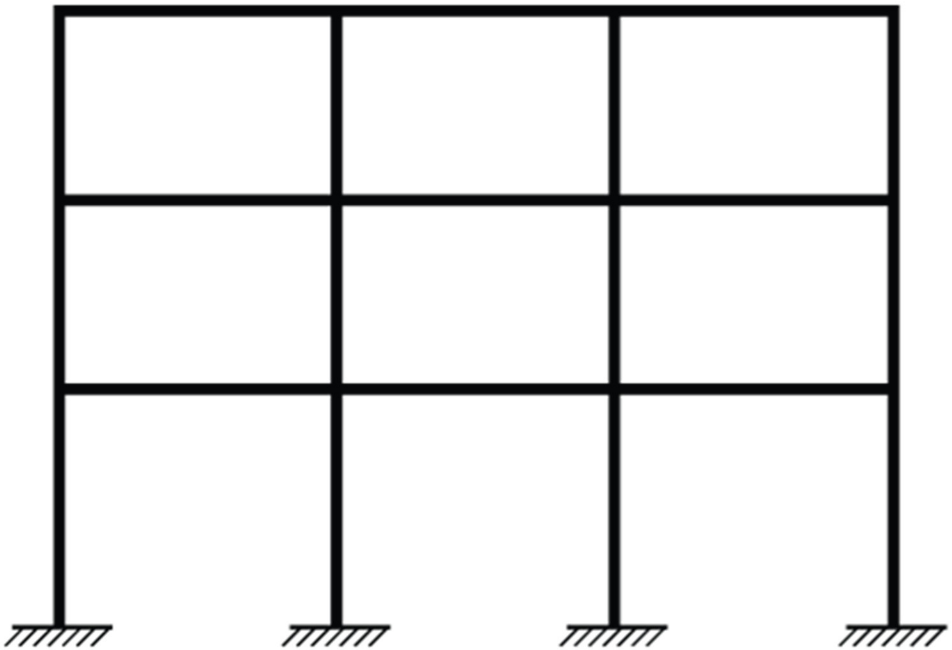
Konzultacije: **srijedom 8:00 — 9:00 sati**
Google Classroom: **qmvjpo6**

Potreba za iterativnim metodama (računalnim iteracijama)



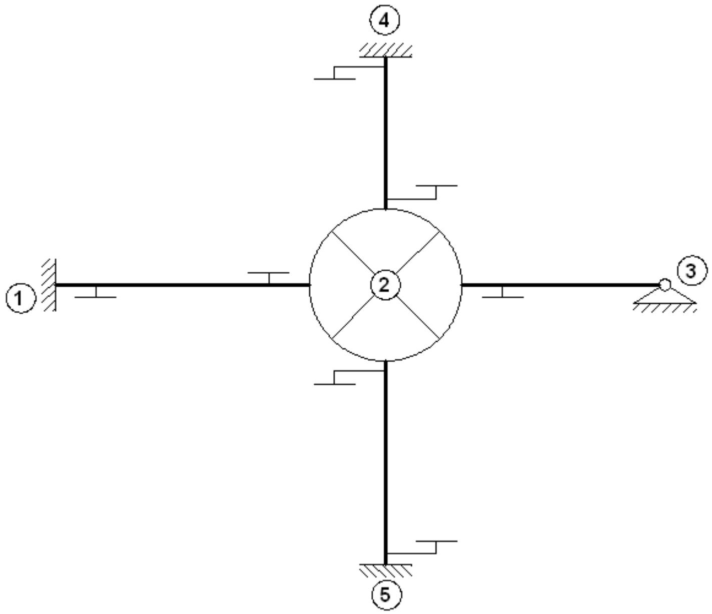
- **27** puta statički neodređen sustav (**MS**)
- **15** stupnjeva slobode (**IMP**)

Iterativne metode

Rješavanje jednadžbi ravnoteže iterativnim postupkom.

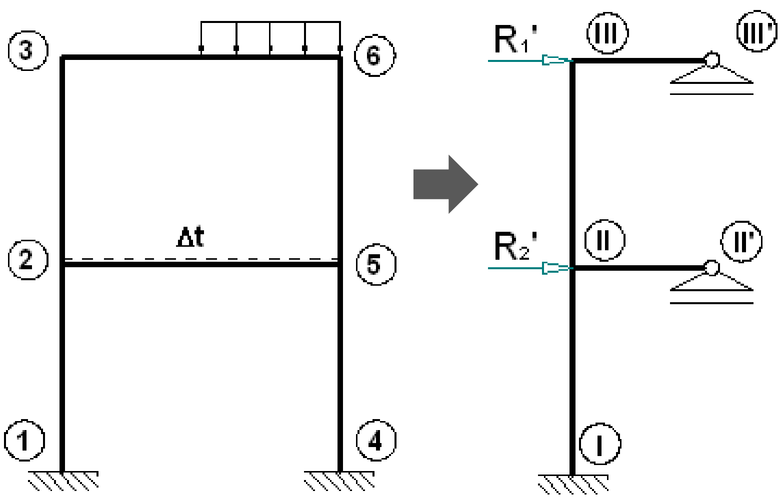
Metoda Crossa

- Pomični i nepomični sustavi
- Bazira se na metodi pomaka



Metoda Csonka-Wernera

- Pomični višekatni. višerasponski sustavi
- Temelji se na postupku poluokvira



Metoda Crossa — Ideja metode

The method of moment distribution is this:

a) Imagine all joints in the structure held so that they cannot rotate and compute the moments at the ends of the member for this condition;
b) at each joint distribute the unbalanced fixed-end moment among the connecting members in proportion to the constant for each member defined as "stiffness";
c) multiply the moment distributed to each member at a joint by the carry-over factor at that end of the member and set this product at the other end of the member;
d) distribute these moments just "carried over";
e) repeat the process until the moments to be carried over are small enough to be neglected; and
f) add all moments - fixed-end moments. distributed moments. moments carried over - at each end of each member to obtain the true moment at the end

Cross. Hardy. **Analysis of Continuous Frames by Distributing Fixed-End Moments**. Transactions of the American Society of Civil Engineers. Vol. 96. No. 1. January 1932. pp. 1-10

Metoda Crossa — Ideja metode

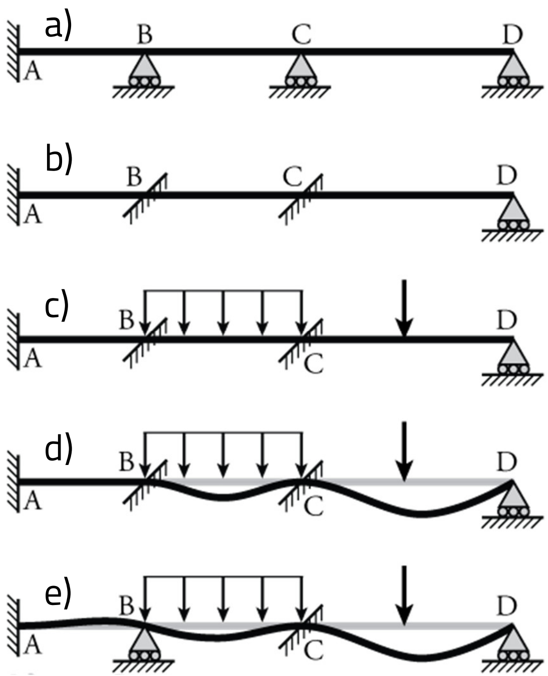
a) Statički sustav

b) Sprječavanje rotacije unutarnjih čvorova

c) Utjecaj opterećenja

d) Izračun **momenata upetosti**

e) Iterativno **otpuštanje rotacije** čvora B

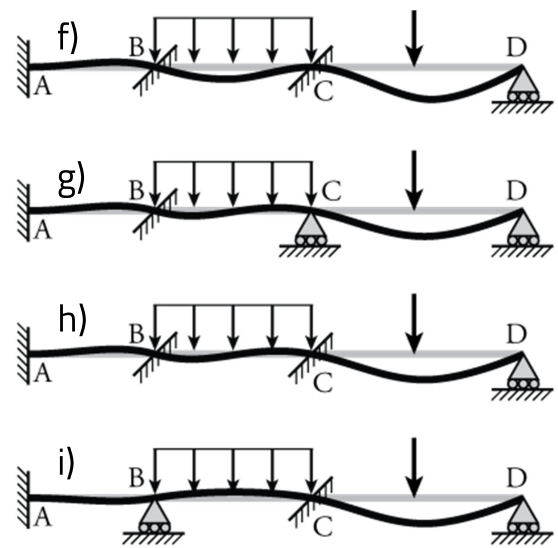


f) Sprječavanje rotacije čvora B i **prijenos utjecaja** na susjedni čvor C

g) Iterativno **otpuštanje rotacije** čvora C

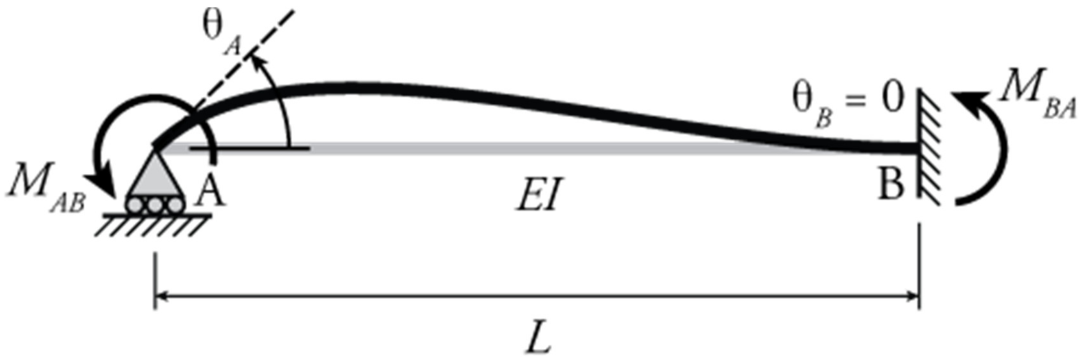
h) Sprječavanje rotacije čvora C i **prijenos utjecaja** na susjedni čvor B

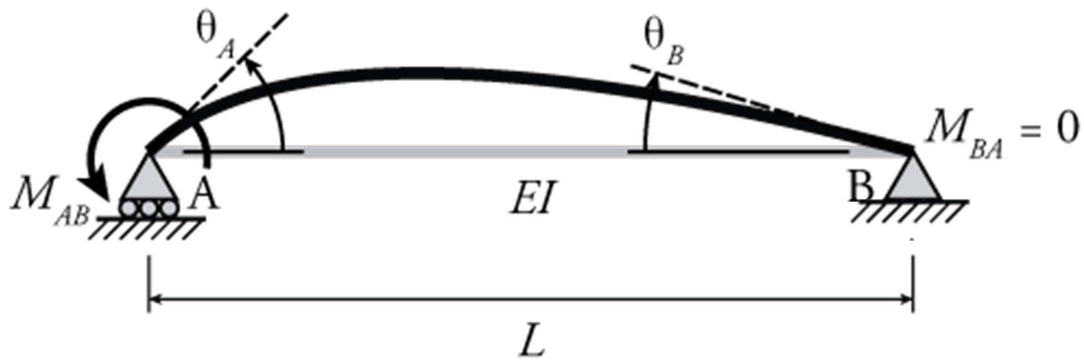
i) Iterativno **otpuštanje rotacije** čvora C



g) Nastavljanje iteracija dok **ostatak prijenosnog momenta** ne bude dovoljno mali (npr. < 0.1 kNm)

Iterativne metode — Metoda Crossa





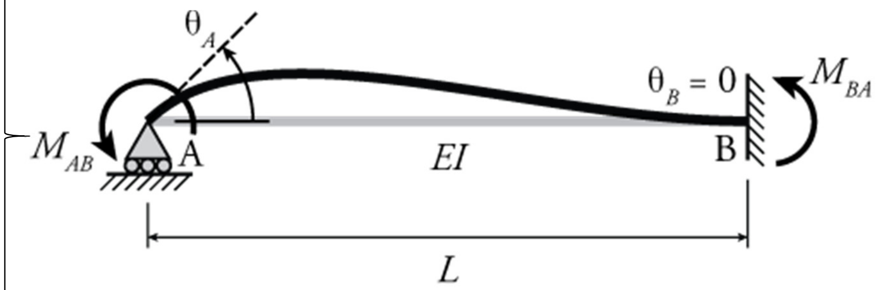
Iterativne metode — Metoda Crossa

$$M_{AB} = \frac{EI}{L}(4\theta_A + 2\theta_B - 6\psi_{AB}) + \bar{M}_{AB}$$

$$M_{AB} = \frac{4EI}{L}\theta_A \qquad k_{AB} = \frac{4EI}{L} \because k = \frac{M}{\theta}$$

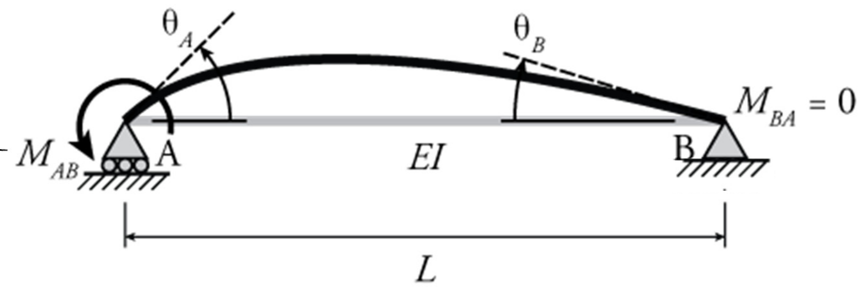
$$M_{BA} = \frac{EI}{L}(4\theta_B + 2\theta_A - 6\psi_{AB}) + \bar{M}_{BA}$$

$$M_{BA} = \frac{2EI}{L}\theta_A$$



$$M_{AB} = \frac{EI}{L}(3\theta_A - 3\psi_{AB}) + \bar{M}_{AB}$$

$$M_{AB} = \frac{3EI}{L}\theta_A \qquad k_{AB} = \frac{3EI}{L} \because k = \frac{M}{\theta}$$



Metoda Crossa — Postupak proračuna

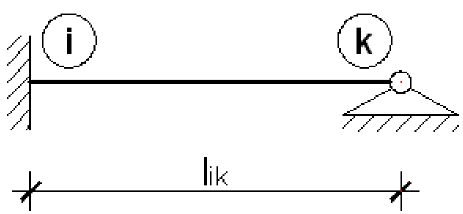
1. Proračun krutosti (presjeka EI i elemenata)
2. **Proračun razdjelnih koeficijenata**
3. Prijenosni koeficijent
4. Momenti upetosti
5. **Iteracija !**
6. Konačni dijagram

1. Proračun krutosti (presjeka EI i elemenata k_{ik})

Stvarna krutost elementa (štapa):

$$k_{ik} = \frac{E_{ik} \cdot I_{ik}}{L_{ik}}$$

Vrijednost proračunske krutosti štapa u slučaju **jednostrano upete grede se umanjuje!**

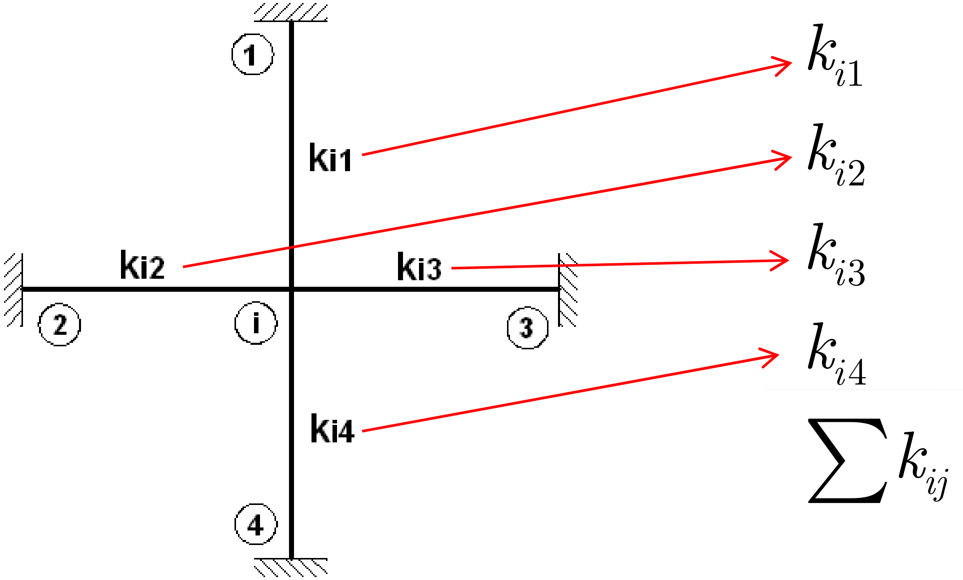


$$k_{ik}' = \frac{3}{4} k_{ik}$$

$$\underbrace{k_{AB} = \frac{3EI}{L}}_{\text{Jednostrano upeti element}} \qquad \underbrace{k_{AB} = \frac{4EI}{L}}_{\text{Obostrano upeti element}}$$

2. Proračun razdjelnih koeficijenata μ_{ik}

Provodi se za sve elemente čvora s nepoznatim vrijednostima momenata. Dobra praksa je zaokruživati razdjelne koeficijente na 3 decimalna mjesta iza zareza.

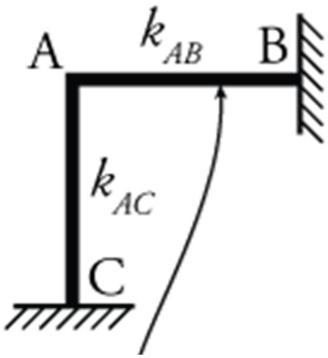


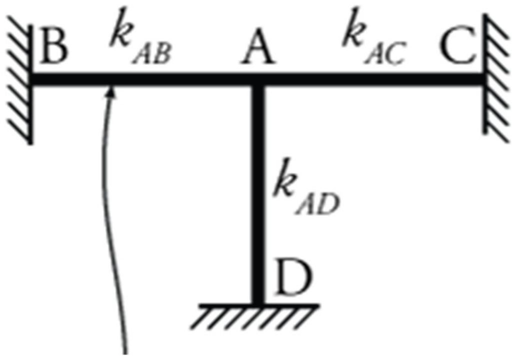
Razdjelni koeficijent:

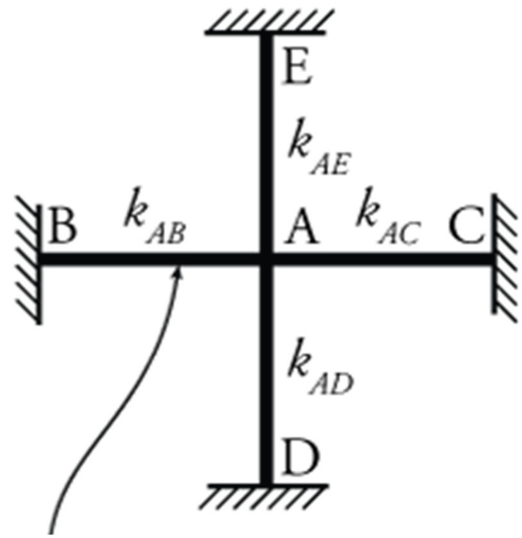
$$\mu_{ik} = \frac{k_i}{\sum k_{ij}}$$

$$\sum \mu_{ik} = 1$$

2. Proračun razdjelnih koeficijenata μ_{ik}







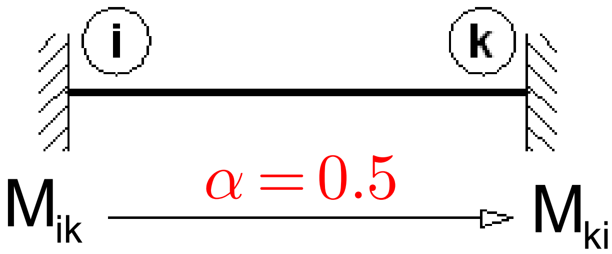
$$\mu_{AB} = \frac{k_{AB}}{k_{AB} + k_{AC}}$$

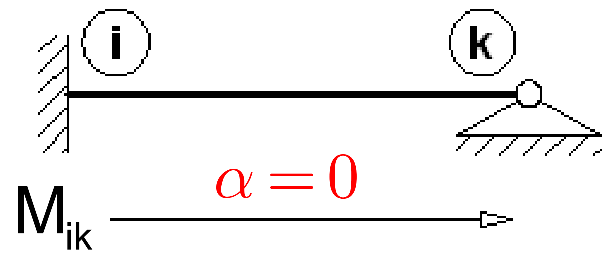
$$\mu_{AB} = \frac{k_{AB}}{k_{AB} + k_{AC} + k_{AD}}$$

$$\mu_{AB} = \frac{k_{AB}}{k_{AB} + k_{AC} + k_{AD} + k_{AE}}$$

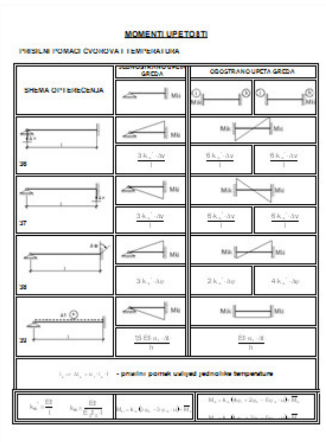
3. Prijenosni koeficijent $\alpha = 0.5$

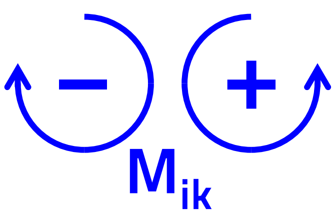
Koristi se pri iteraciji samo za obostrano upete grede





4. Momenti upetosti (tablice!)





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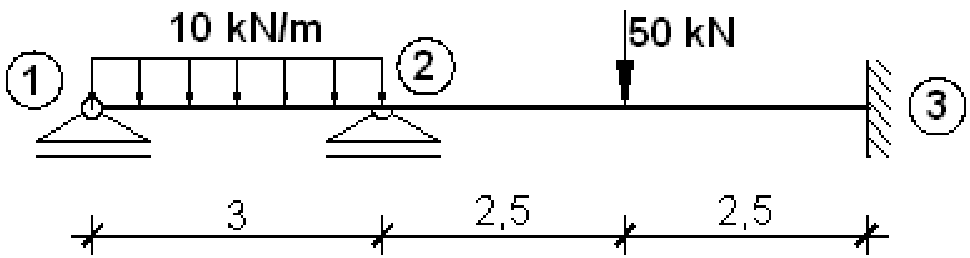
5. Postupak iteracije

1. Iteracija započinje u čvoru sa **najvećom apsolutnom vrijednosti** momenta upetosti M
2. Momentu M se mijenja predznak u **–M** radi ravnoteže
3. Moment **–M** se **množi s razdjelnim koeficijentima** elemenata promatranog čvora i zapisuje se svaka vrijednost i podvlači kao znak da je uspostavljena ravnoteža čvora
4. Vrijednosti momenata na **obostrano upetim elemetima** se na drugi kraj elementa prenose **prijenosnim koeficijentom $\alpha = 0.5$**
5. Postupak se nastavlja u čvoru sa **sljedećim najvećim apsolutnim momentom**
6. Postupak iteracije je gotov kada se postigne ravnoteža u svim čvorovima (ostatak momenta **< 0.1 kNm**)

5. Postupak iteracije — Primjer

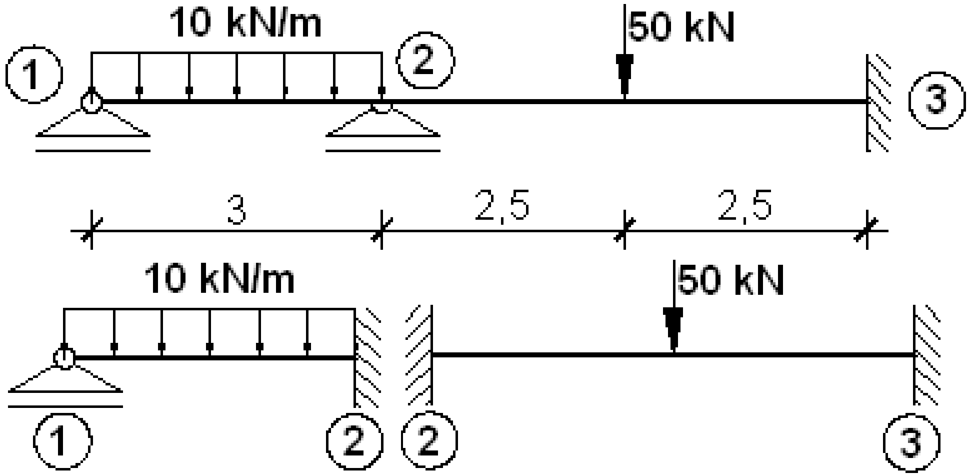


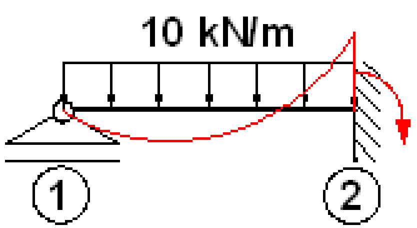
$$EI = 20\,250\ \text{kNm}^2$$

$$k_{12} = \frac{3}{4} \cdot \frac{20\,250}{3} = 5\,062.5\ \text{kNm}; \qquad k_{23} = \frac{20\,250}{5} = 4\,050\ \text{kNm}$$

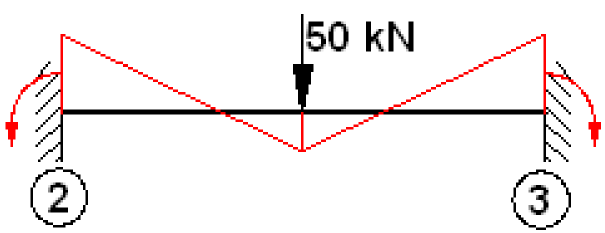
Čvor	Element	k_i [kNm]	Σk_i [kNm]	μ_i
2	2-1	**5 062.5**	9 112.5	**0.556**
	2-3	**4 050**		**0.444**

$$\sum = 1.0$$





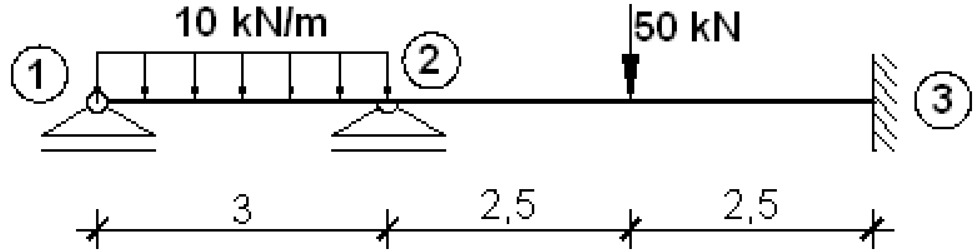
$$\bar{M}_{21} = -\frac{qL^2}{8} = -11.25 \text{ kNm}$$



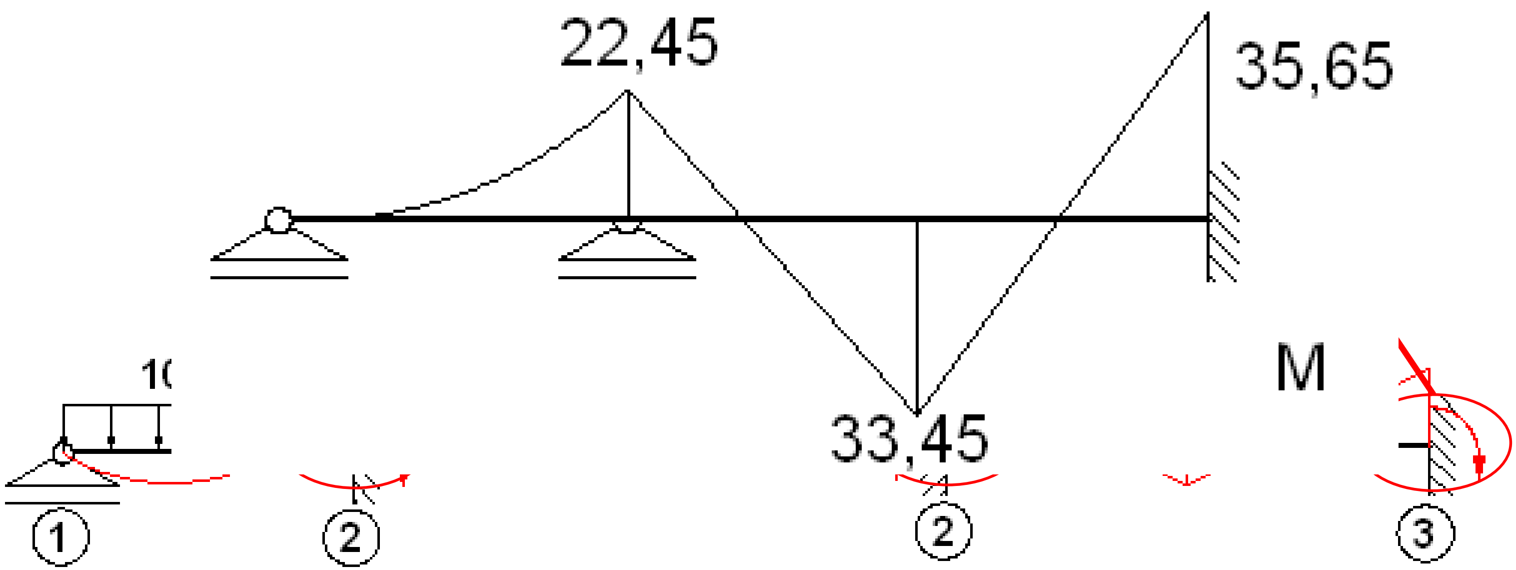
$$\bar{M}_{23} = \frac{PL}{8} = +31.25 \text{ kNm};$$

$$\bar{M}_{32} = -31.25 \text{ kNm}$$

Štap	μ_i
2-1	0.556
2-3	0.444



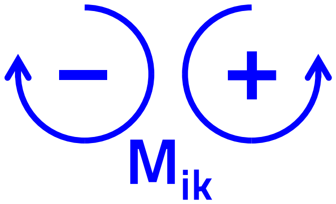
$$\Rightarrow M_2 = -11.25 + 31.25 = 20 \text{ kNm} \Rightarrow -20 \text{ kNm}$$



6. Konačni dijagram

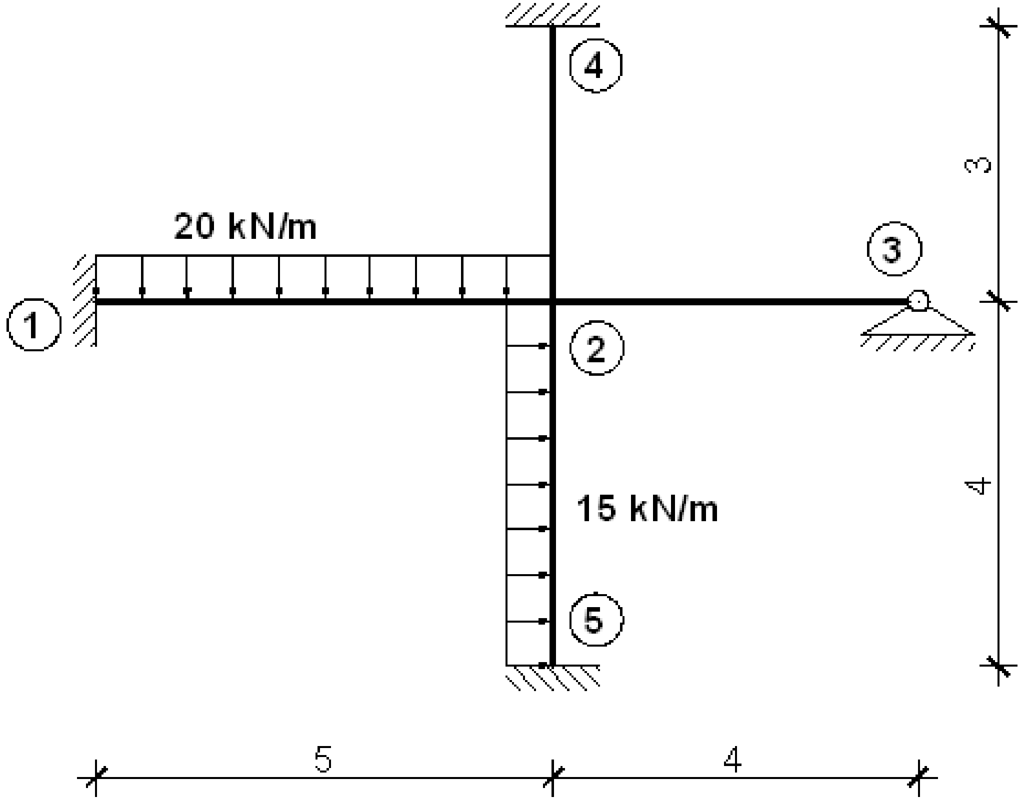
Momenti savijanja na krajevima elemenata:

- zbrajanje svih vrijednosti iteracija za pojedini kraj promatranog elementa.
- crtanje dijagrama momenata savijanja jednako kao u metodi pomaka – predznak određuje vrtnju momenta.



Zadatak #1

Za prikazani sustav metodom Crossa odrediti dijagram momenata savijanja.



STUP: b/h = 40/40 cm

GREDA: b/h = 40/50 cm

E = 30 GPa

1. Proračun krutosti

$$\text{Stup:} \quad EI = 64\,000 \text{ kNm}^2$$

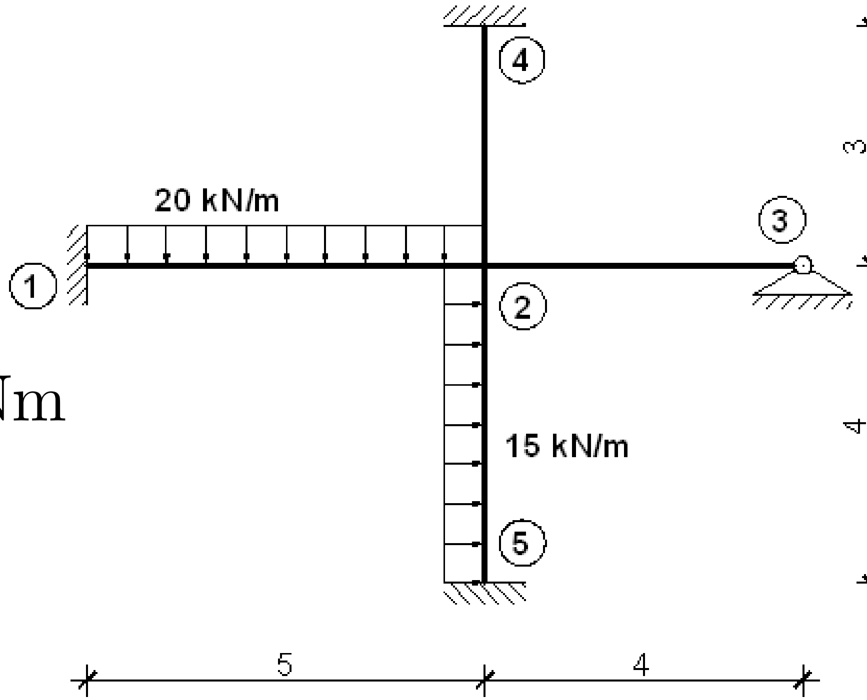
$$\text{Greda:} \quad EI = 125\,000 \text{ kNm}^2$$

$$k_{12} = \frac{125\,000}{5} = 25\,000 \text{ kNm}$$

$$k_{23} = \frac{3}{4} \cdot \frac{125\,000}{4} = 23\,437.5 \text{ kNm}$$

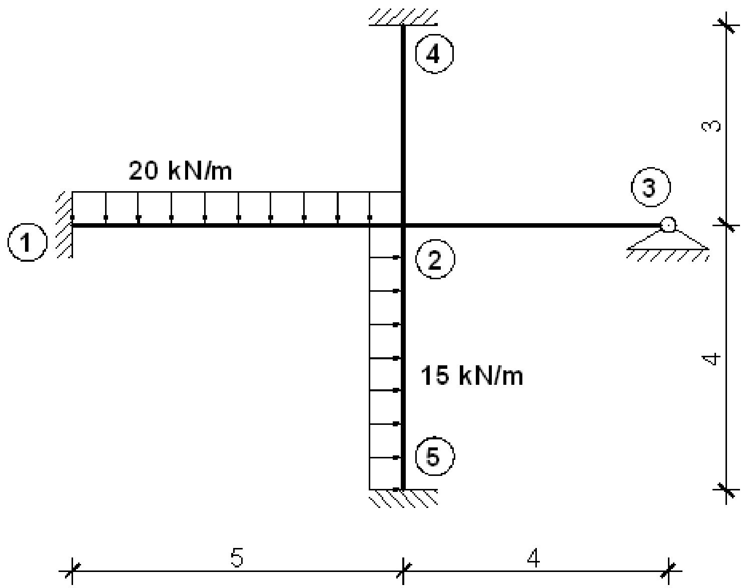
$$k_{24} = \frac{64\,000}{3} = 21\,333.33 \text{ kNm}$$

$$k_{25} = \frac{64\,000}{4} = 16\,000 \text{ kNm}$$



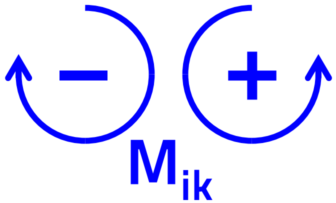
2. Proračun razdjelnih koeficijenata

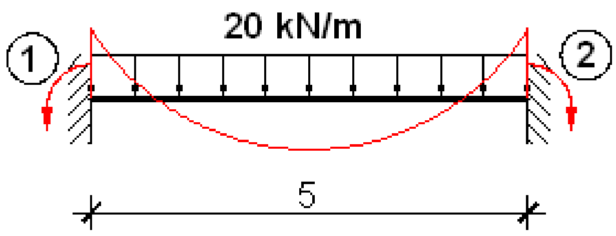
Čvor	Element	k_i [kNm]	Σk_i [kNm]	μ_i
2	2-1	**25 000**	85 770.83	**0.291**
	2-3	**23 437.5**		**0.273**
	2-4	**21 333.33**		**0.249**
	2-5	**16 000**		**0.187**
				1.00



3. Prijenosni koeficijent. α = 0.5

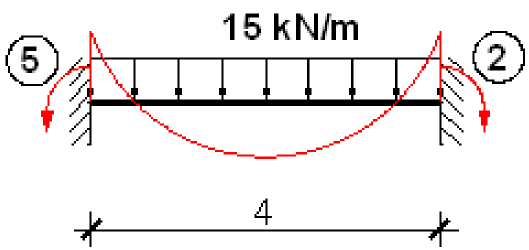
4. Momenti upetosti





$$\bar{M}_{12} = \frac{qL^2}{12} = +41.67\ \text{kNm}$$

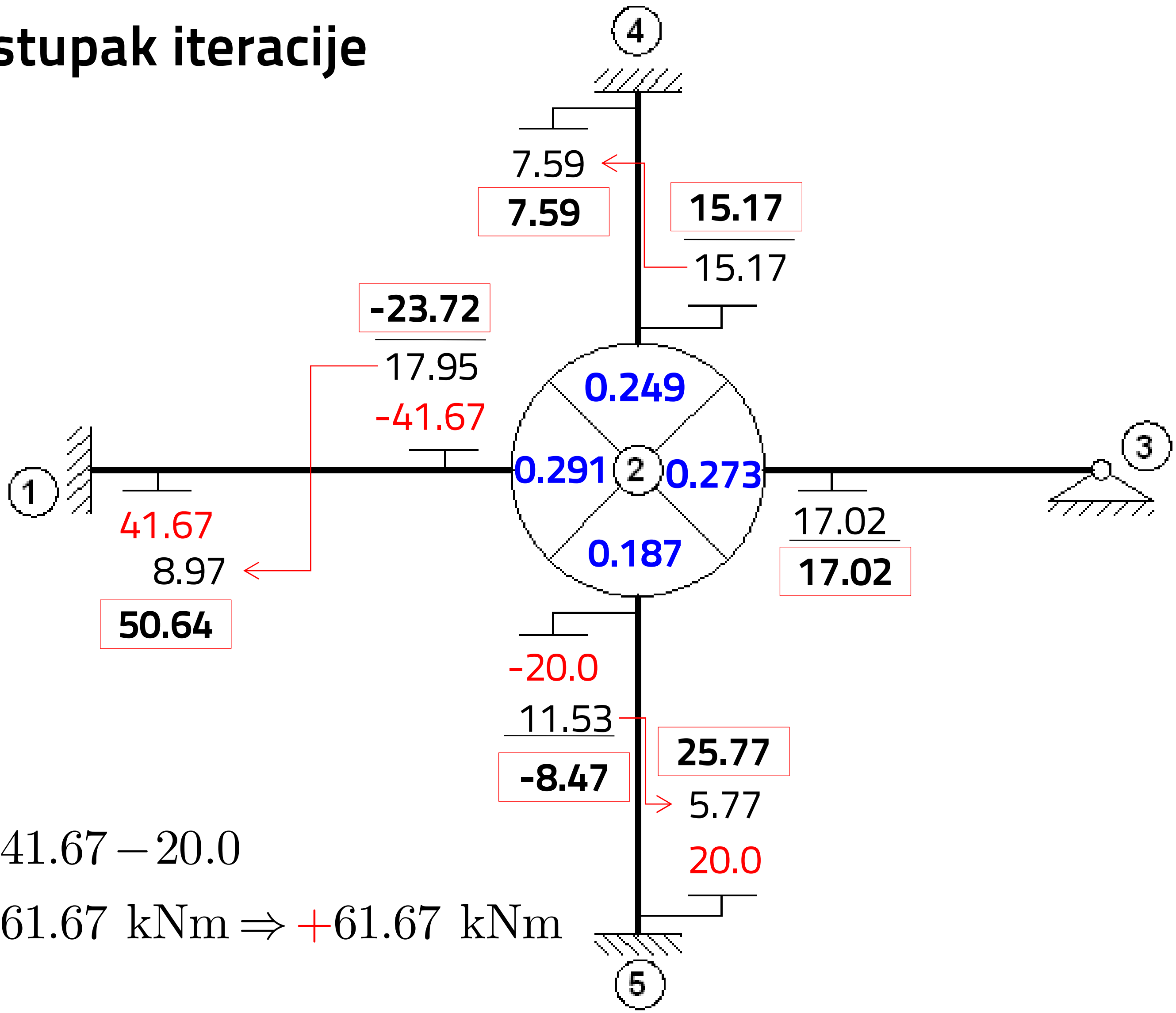
$$\bar{M}_{21} = -41.67\ \text{kNm}$$



$$\bar{M}_{52} = \frac{qL^2}{12} = +20.0\ \text{kNm}$$

$$\bar{M}_{25} = -20.0\ \text{kNm}$$

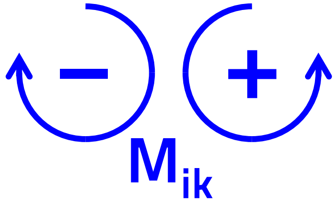
5. Postupak iteracije

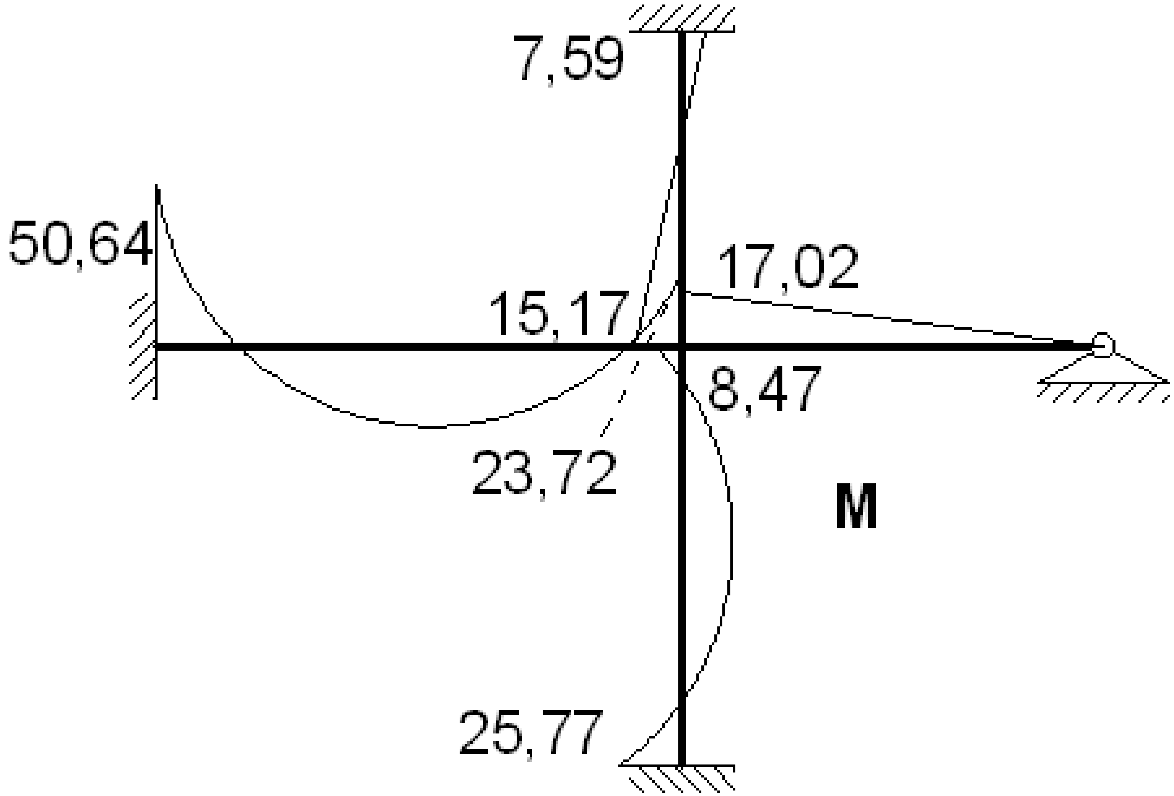


$$M_2 = -41.67 - 20.0$$

$$M_2 = -61.67 \text{ kNm} \Rightarrow +61.67 \text{ kNm}$$

6. Konačni momentni dijagram

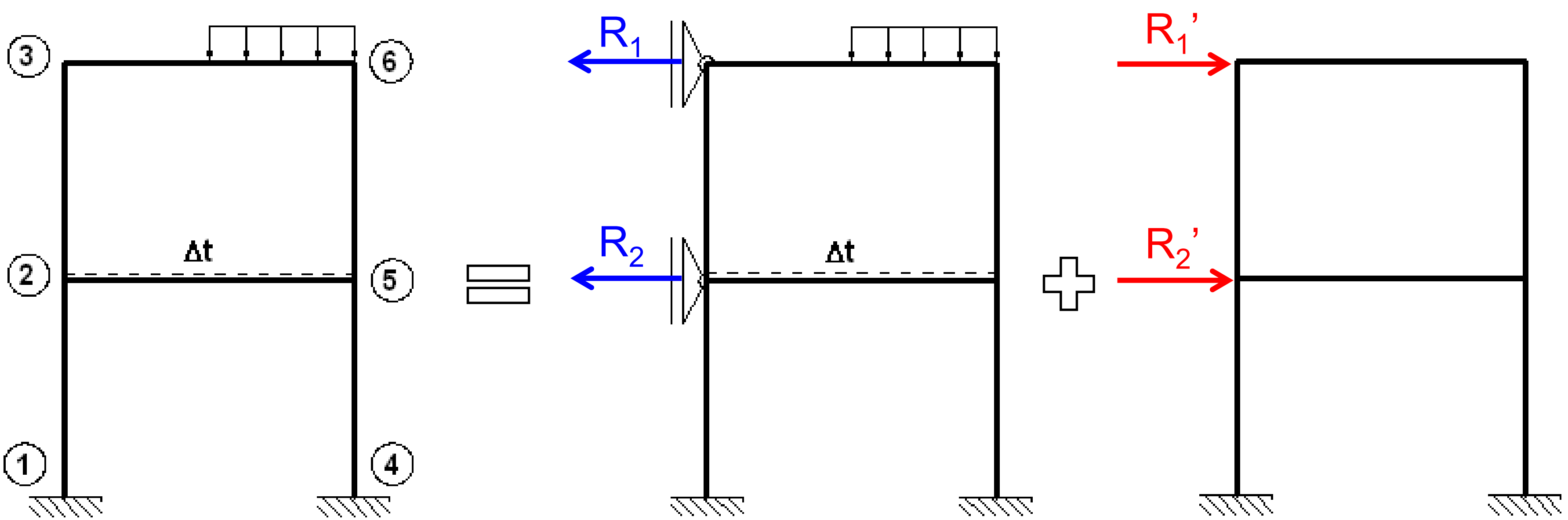




Metoda Csonka-Wernera (CW)

Primjenjuje se kod **pomičnih okvirnih višekatnih. višerasponskih konstrukcija**. pri čemu se sustav svodi na pojednostavljeni poluokvir.

1. FAZA **2. FAZA**



Konačni dijagram M = **CROSS M_I** + **CSONKA-WERNER M_{II}**

Ukoliko je zadan **pomični sustav** sa opterećenjem koje **ne djeluje u čvorovima** konstrukcije. proračun se provodi u dvije faze:

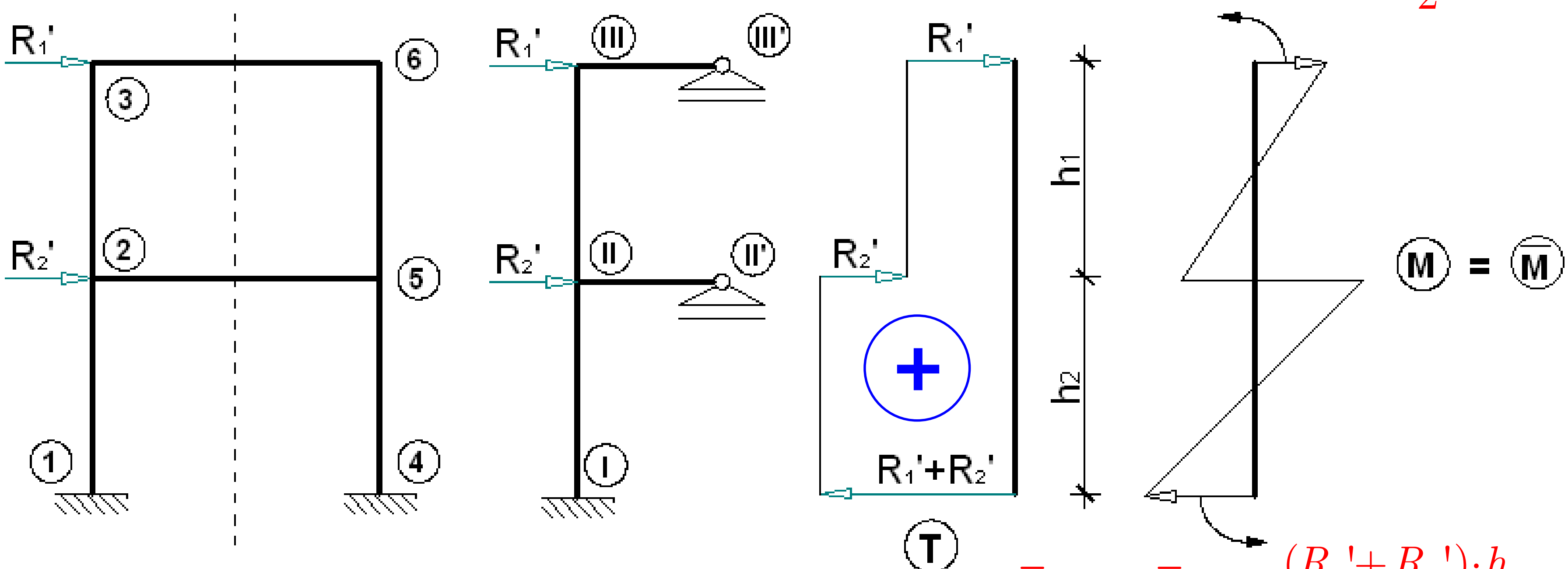
1. FAZA (Cross) – obuhvaća **dodavanje pridržanja** (pomični ležajevi okomito na smjer mogućih gibanja!) u nivoima međukatnih konstrukcija zbog onemogućavanja pomaka zadane pomične konstrukcije. Tako pridržana. nepomična konstrukcije se dalje proračunava postupkom **Crossa**. pri čemu se kao rezultat dobiva momentni dijagram $\mathbf{M_{I}}$.

2. FAZA (Csonka-Werner) – reakcije (R) koje su dobivene u dodanim pridržanjima nanose se **kao opterećenje** na pomičnu konstrukciju pri čemu su reakcije istih vrijednosti, ali suprotnog smjera (R'). Pomoću **vrijednosti reakcija R'** (ili zadanih sila u nivoima međukatnih konstrukcija) određuje se **dijagram poprečnih sila na poluokviru.**

Određivanje momenata upetosti na poluokviru (metoda CW)

Iz dijagrama **poprečnih sila** određuje se **dijagram momenata** koji daje **vrijednosti momenata upetosti** potrebne za iteraciju po metodi Csonka-Wernera.

$$\bar{M}_{II,III} = \bar{M}_{III,II} = \frac{R_1' \cdot h_1}{2}$$



$$\bar{M}_{I,II} = \bar{M}_{II,I} = \frac{(R_1' + R_2') \cdot h_2}{2}$$

Metoda Csonka-Wernera (CW) — Postupak proračuna

1. **Proračun krutosti (!)**
2. **Proračun razdjelnih koeficijenata (!)**
3. **Prijenosni koeficijent (!)**
4. **Momenti upetosti (!)**
5. **Iteracija na poluokviru (!)**
6. **Prijenos momenata sa poluokvira na početni sustav (!)**
7. Uravnoteženje po Crossu
8. **Kontrola poprečne sile (!)**
9. Konačni M dijagram

1. Proračun krutosti

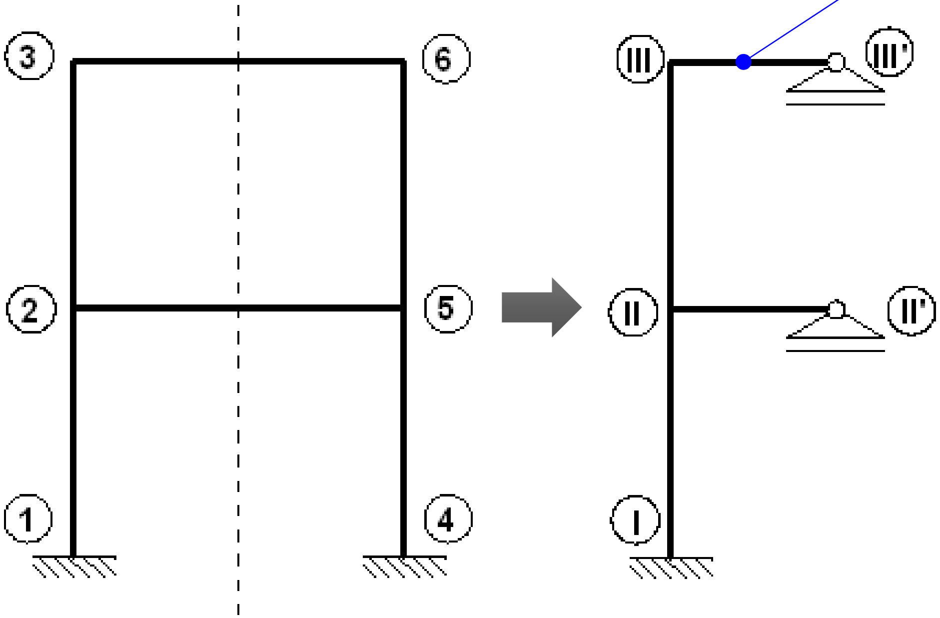
$$k_{ik} = \frac{E_{ik} \cdot I_{ik}}{L_{ik}}$$

$$k_{ik}' = \frac{3}{4} k_{ik}$$

Redukcija krutosti samo za jednostrano upete elemente

$$M_{ik} = 3\, k_{ik}\, \theta_i = 12 \frac{EI}{L} \theta_i$$

$$k_{ik} = \frac{E(2I)}{\frac{L}{2}} = 4\frac{EI}{L}$$



Krutosti za originalni okvir:

Stupovi: k_{12}; k_{23}; k_{45}; k_{56}

Grede: k_{25}; k_{36}

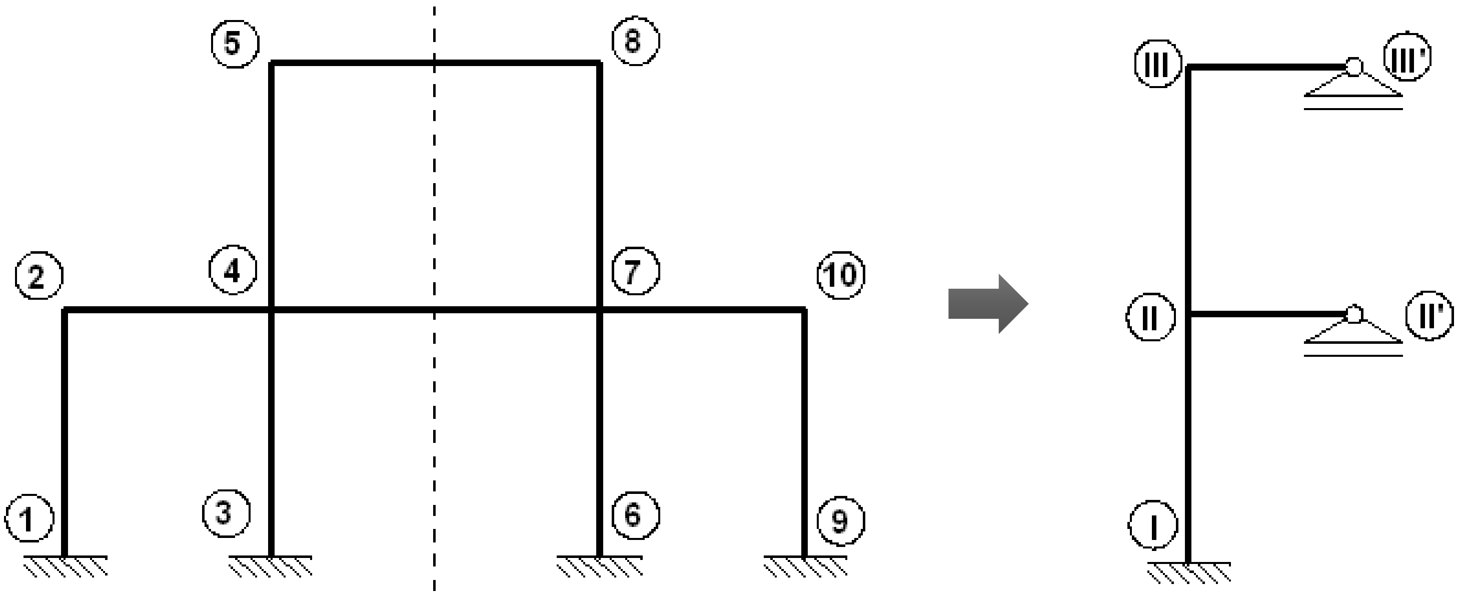
Krutosti za zamjenski poluokvir:

Stupovi: $k_{I.II} = k_{12} + k_{45}$

$k_{II.III} = k_{23} + k_{56}$

Grede: $k_{II.II'} = 12 \cdot k_{25}$

$k_{III.III'} = 12 \cdot k_{36}$



Krutosti za zamjenski poluokvir:

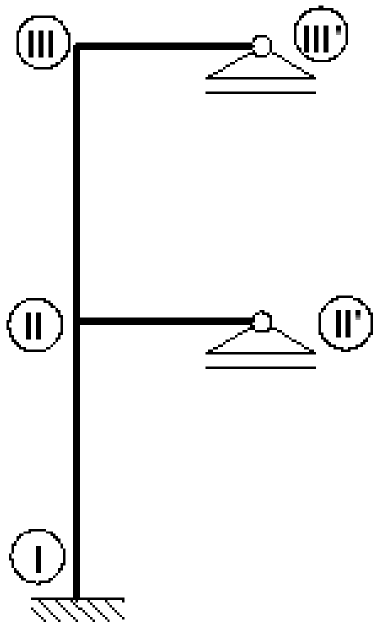
Stupovi: $k_{I.II} = k_{12} + k_{34} + k_{67} + k_{910}$

$k_{II.III} = k_{45} + k_{78}$

Grede: $k_{II.II'} = 12 \cdot (k_{24} + k_{47} + k_{710})$

$k_{III.III'} = 12 \cdot k_{58}$

2. Proračun razdjelnih koeficijenata



Razdjelni koeficijent predstavlja omjer krutosti pojedinog štapa promatranog čvora i **sume svih krutosti** štapova u čvoru. Suma razdjelnih koeficijenata u pojedinom čvoru mora dati jediničnu vrijednost.

$$\mu_i = \frac{k_i}{\sum k_i}$$

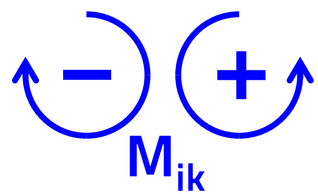
Čvor	Element	k_i [kNm]	Σk_i [kNm]	μ_i	$\Sigma\mu_i$
II	II-I II-II' II-III				1.0
III	III-II III-III'				1.0

3. Prijenosni koeficijent. $\alpha = -1$ (2. faza proračuna po CW)

Koristi se pri iteraciji i predstavlja vrijednost kojom se množi moment kada se s jednog kraja štapa prenosi na drugi kraj (kod poluokvira prijenosni koeficijent se koristi **samo za stupove**).

4. Momenti upetosti

Na poluokviru određuju se **iz dijagrama poprečnih sila**. kako je prikazano ranije.

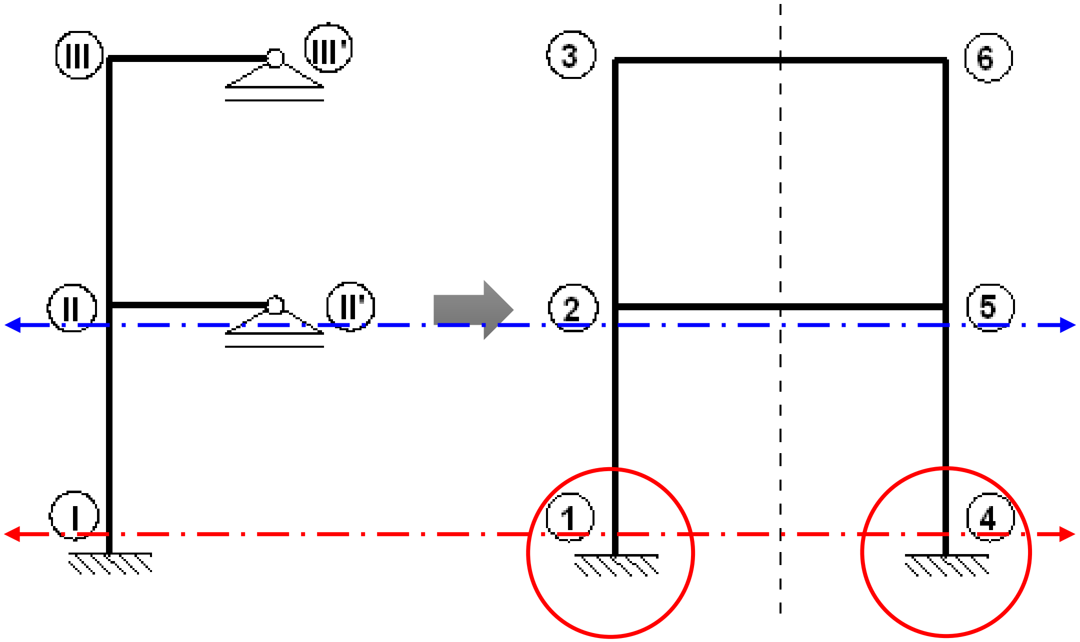


5. Iteracija na poluokviru

Iteraciju započinjemo u čvoru sa najvećom apsolutnom vrijednosti momenta. Momenti iste vrijednosti. ali suprotnog predznaka množe se sa razdjelnim koeficijentima. prenose sa prijenosnim koeficijentima na druge krajeve stupova. Postupak se nastavlja sa sljedećim maksimalnim apsolutnim momentom.

Postupak je gotov kada se **postigne ravnoteža u svakom čvoru**!

6. Prijenos momenata s poluokvira na početni sustav



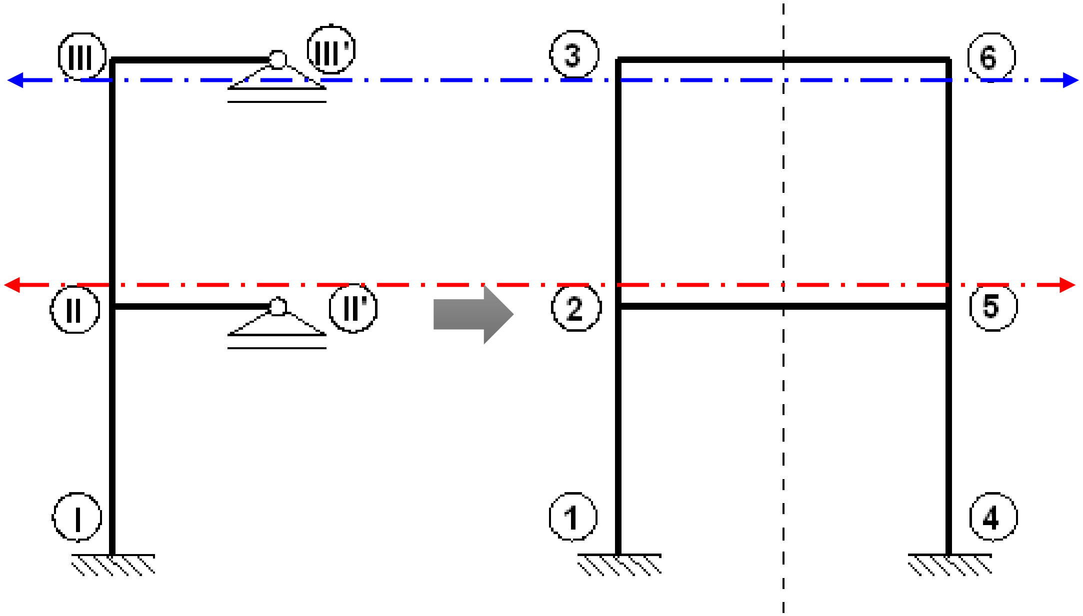
Momenti na poluokviru:

Stupovi: $M_{I.II}$; $M_{II.I}$;
$M_{II.III}$; $M_{III.II}$

Grede: $M_{II.II'}$; $M_{III.III'}$

$$M_{12} = \frac{k_{12}}{k_{12} + k_{45}} \cdot M_{I,II} \qquad M_{45} = \frac{k_{12}}{k_{12} + k_{45}} \cdot M_{II,I}$$

$$M_{21} = \frac{k_{12}}{k_{12} + k_{45}} \cdot M_{II,I} \qquad M_{54} = \frac{k_{45}}{k_{12} + k_{45}} \cdot M_{II,I}$$



$$M_{23} = \frac{k_{23}}{k_{23} + k_{56}} \cdot M_{II,III} \qquad M_{56} = \frac{k_{56}}{k_{23} + k_{56}} \cdot M_{II,III} \qquad M_{25} = M_{52} = \frac{M_{II,II'}}{2}$$

$$M_{32} = \frac{k_{23}}{k_{23} + k_{56}} \cdot M_{III,II} \qquad M_{65} = \frac{k_{56}}{k_{23} + k_{56}} \cdot M_{III,II} \qquad M_{36} = M_{63} = \frac{M_{III,III'}}{2}$$

7. Uravnoteženje po Crossu

Nakon iteracije po Csonka-Werneru i vraćanja vrijednosti na početni sustav. potrebno je napraviti uravnoteženje po Crossu. Taj momentni dijagram označava se sa $\mathbf{M_{II}}$.

8. Kontrola poprečne sile (β)

Iz $\mathbf{M_{II}}$ dijagrama se **pomoću diferencijalnih odnosa** dobivaju vrijednosti poprečnih sila.

Od dobivenih **poprečnih sila** se određuju sile koje djeluju u **nivoima međukatnih konstrukcija.** Usporedbom tih vrijednosti sa početnim dodanim reakcijama dobiva se **popravni koeficijent β**.

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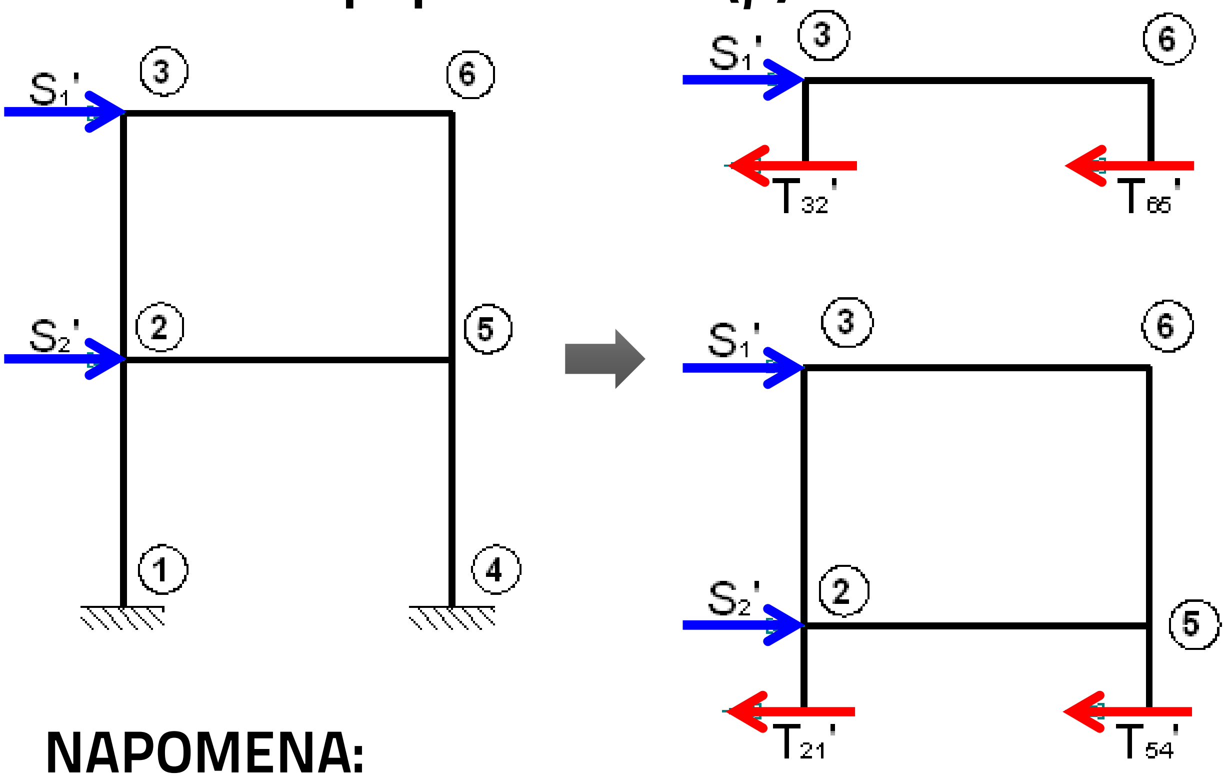
8. Kontrola poprečne sile (β)

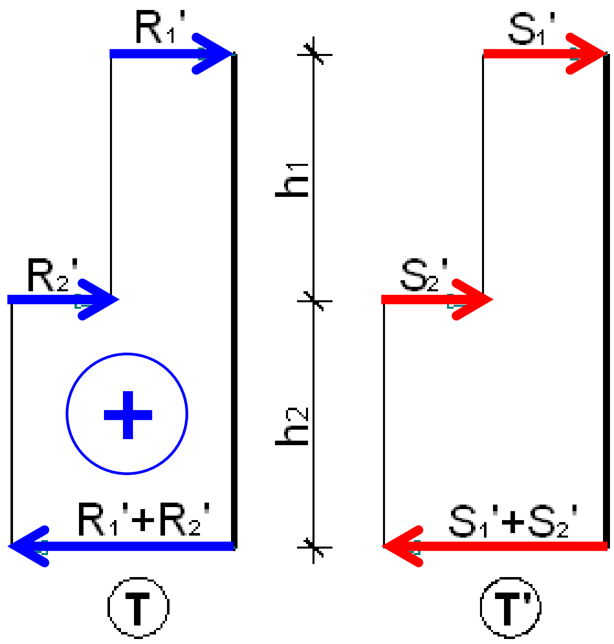


$$S_1' = V_{32}' + V_{65}'$$

$$S_2' = V_{21}' + V_{54}' - S_1'$$

NAPOMENA:

Uvijek koristiti ovu konvenciju predznaka (poprečne sile određene iz M_{II} dijagama <u>unositi sa predznakom</u> u jednadžbe za određivanje **S_1'** i **S_2'**).



$$\beta = \frac{\sum_{i=1}^{n} |R_i'| \cdot h_i}{\sum_{i=1}^{n} |S_i'| \cdot h_i}$$

Popravni koeficijent

$$\beta = \frac{|R_1'| \cdot h_1 + |R_2'| \cdot h_2}{|S_1'| \cdot h_1 + |S_2'| \cdot h_2}$$

9. Konačni momentni dijagram

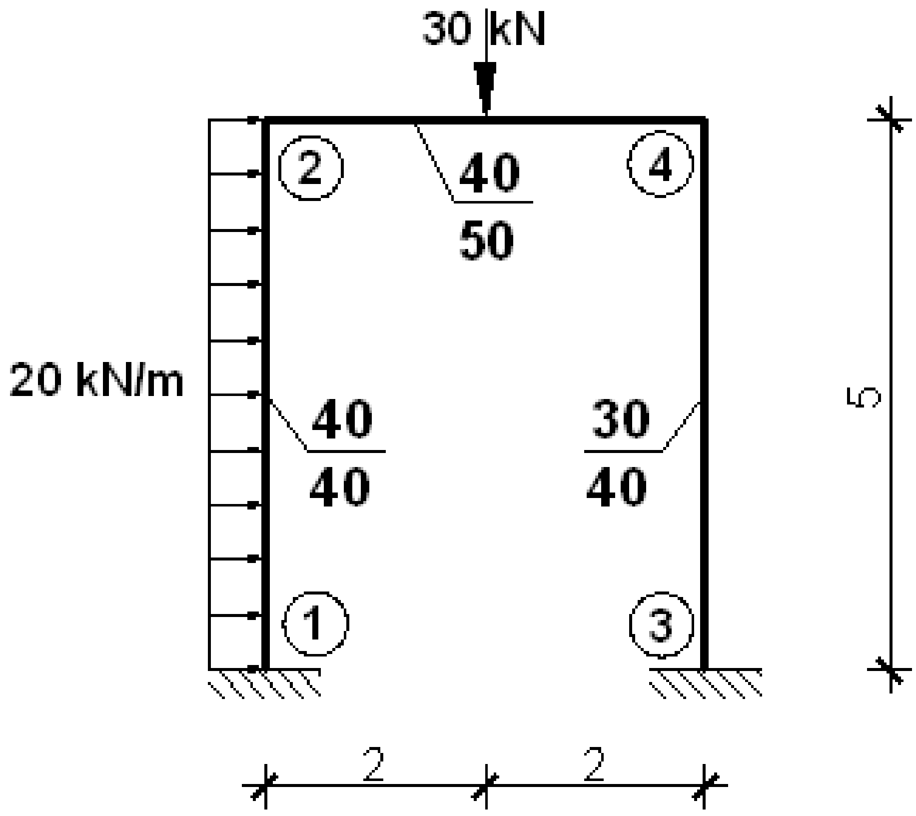
$$M = M_I + \beta \cdot M_{II}$$

M_I - dijagram dobiven metodom Crossa na pridržanom sustavu.

M_{II} - dijagram dobiven uravnoteženjem po Crossu na pomičnom sustavu

Zadatak #2

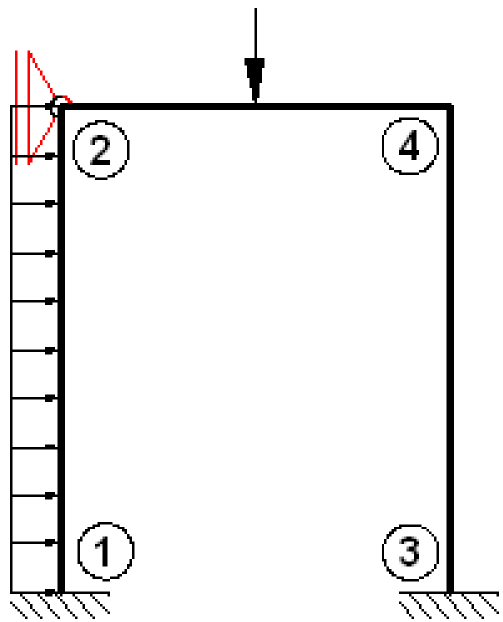
Kombinacijom metoda Crossa i Csonka-Wernera odrediti dijagram momenata savijanja.



$$E = 30 \text{ GPa}$$

1. Proračun po metodi Crossa (1. faza proračuna)

1.1 Pridržani sustav



1.2 Proračun krutosti

$$EI_{12} = 64\,000\ \text{kNm}^2; \qquad EI_{24} = 125\,000\ \text{kNm}^2$$

$$EI_{34} = 48\,000\ \text{kNm}^2$$

1.3 Razdjelni koeficijenti

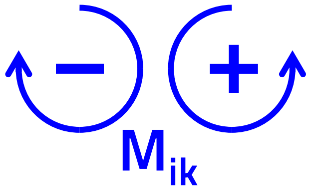
$$k_{12} = \frac{64\,000}{5} = 12\,800 \text{ kNm}$$

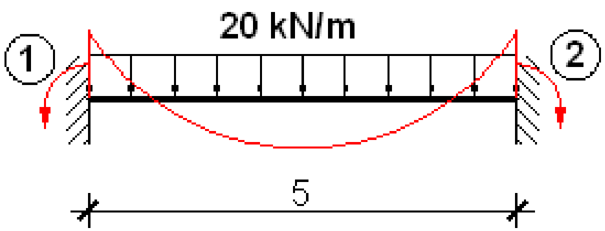
$$k_{24} = \frac{125\,000}{4} = 31\,250 \text{ kNm}; \qquad k_{34} = \frac{48\,000}{5} = 9\,600 \text{ kNm}$$

Čvor	Element	k_i [kNm]	Σk_i [kNm]	μ_i	$\Sigma \mu_i$
2	2-1	**12 800**	44 050	**0.291**	1.0
	2-4	**31 250**		**0.709**	
4	4-2	**31 250**	40 850	**0.765**	1.0
	4-3	**9 600**		**0.235**	

1.4 Prijenosni koeficijent. $\alpha = 0.5$ (1. faza proračuna po Crossu)

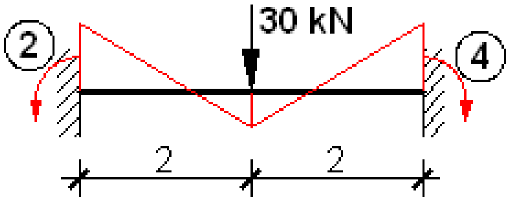
1.5 Momenti upetosti





$$\bar{M}_{12} = \frac{qL^2}{12} = +41.67\ \text{kNm}$$

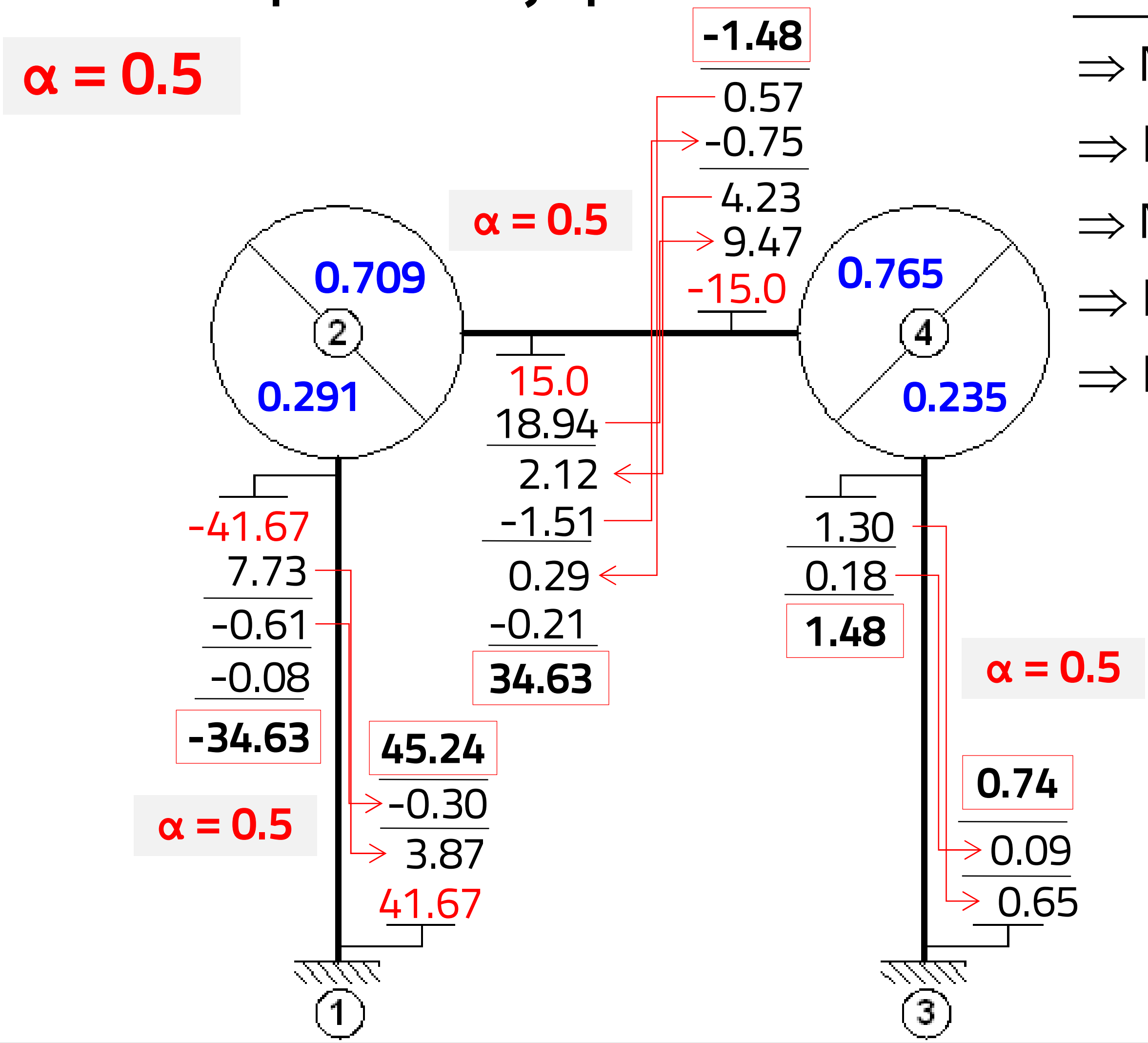
$$\bar{M}_{21} = -41.67\ \text{kNm}$$



$$\bar{M}_{24} = \frac{PL}{8} = +15.0\ \text{kNm}$$

$$\bar{M}_{42} = -15.0\ \text{kNm}$$

1.6 Postupak iteracije po Crossu



Koraci iteracija:

$\Rightarrow M_2 = -26.67 \Rightarrow +26.67$ kNm

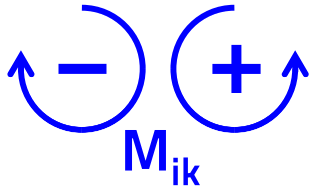
$\Rightarrow M_4 = -5.53 \Rightarrow +5.53$ kNm

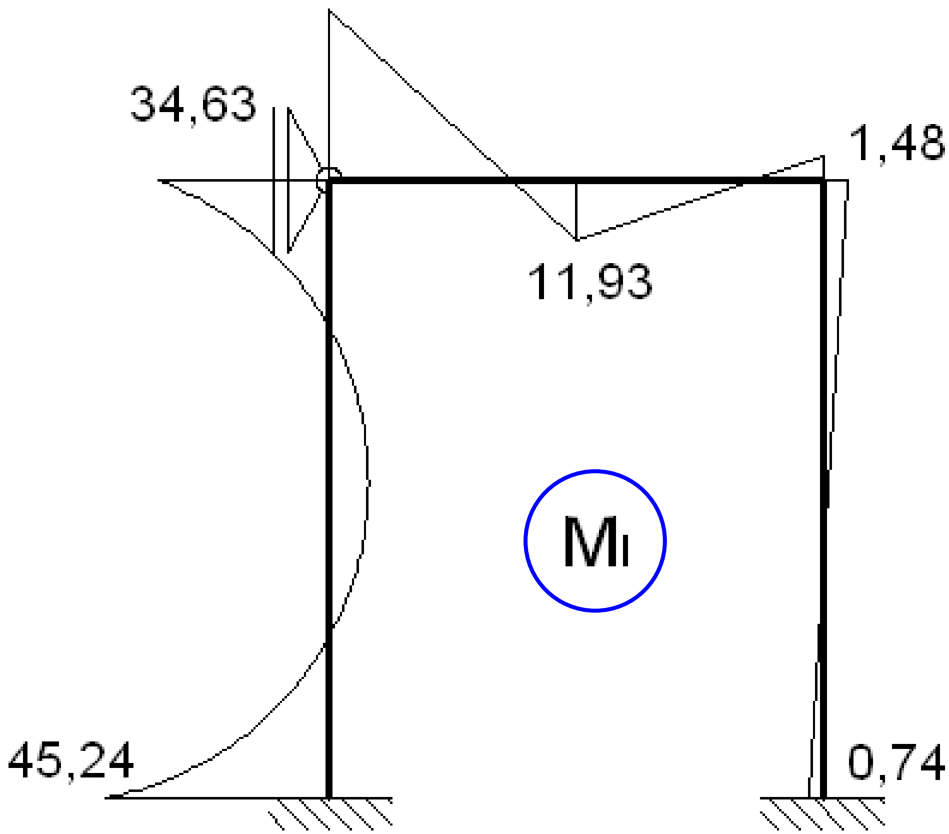
$\Rightarrow M_2 = +2.12 \Rightarrow -2.12$ kNm

$\Rightarrow M_4 = -0.75 \Rightarrow +0.75$ kNm

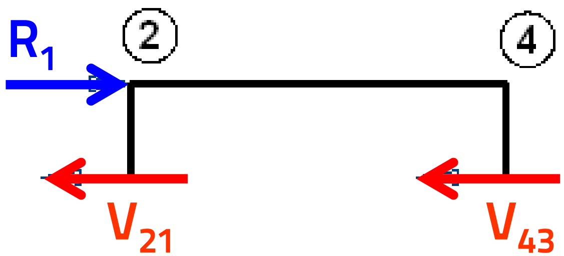
$\Rightarrow M_2 = +0.29 \Rightarrow -0.29$ kNm

1.7 M_I dijagram





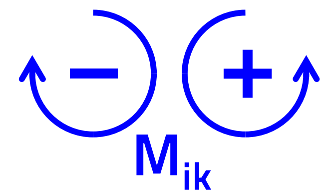
1.8 Proračun reakcija u osloncima pridržanja

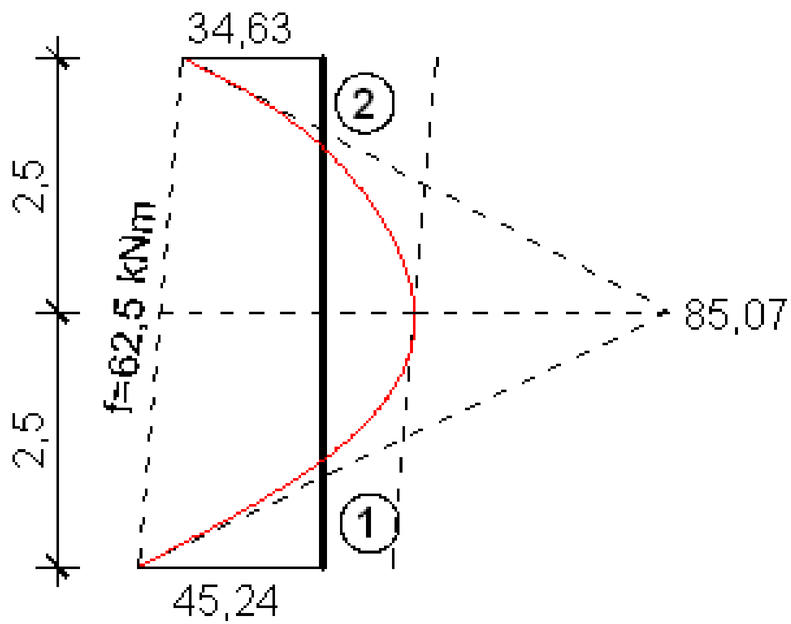


$$+R_1 = +V_{21} + V_{43}$$

Poprečne sile iz **M_I** dijagrama određujemo pomoću diferencijalnih odnosa!

$$V_{ik} = \frac{\pm M_i \pm M_k}{L_{ik}}$$





$$V_{21} = \frac{-34.63 - 85.07}{2.5} = -47.88 \text{ kN}$$

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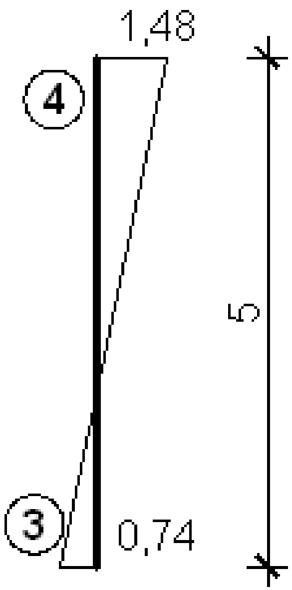
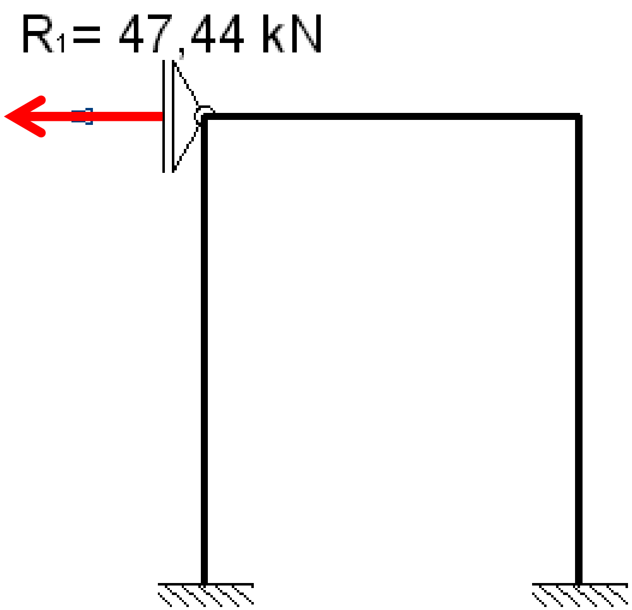


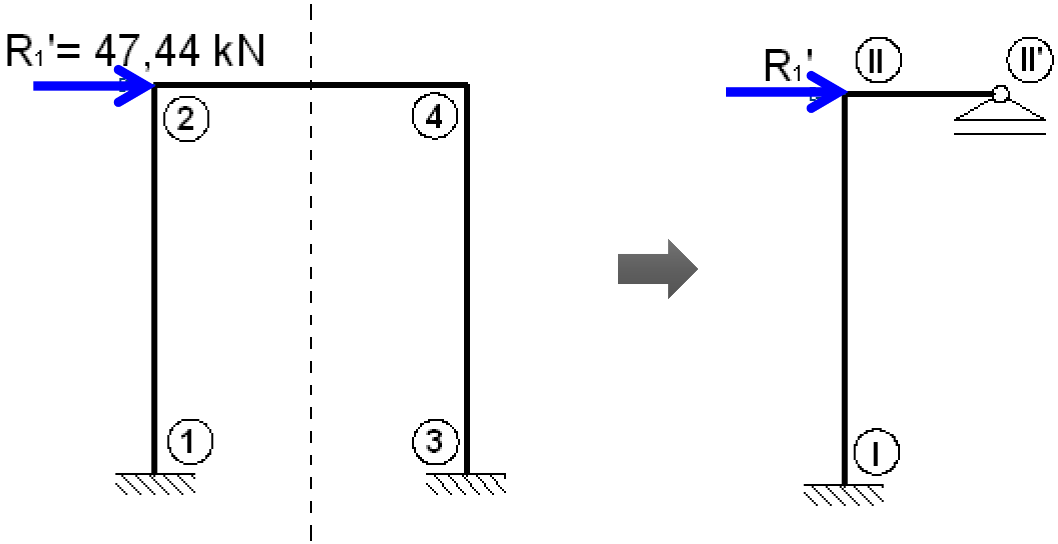
$$V_{43} = \frac{+1.48 + 0.74}{5} = +0.44 \text{ kN}$$

$$R_1 = V_{21} + V_{43} = -47.88 + 0.44 = -47.44 \text{ kN}$$



2. Proračun po metodi Csonka-Wernera (2. faza proračuna)

2.1 Statički sustav



2.2 Proračun krutosti

Krutosti za poluokvir:

Stup: $k_{I.II} = k_{12} + k_{34} = 12\,800 + 9\,600 = \mathbf{22\,400\ kNm}$

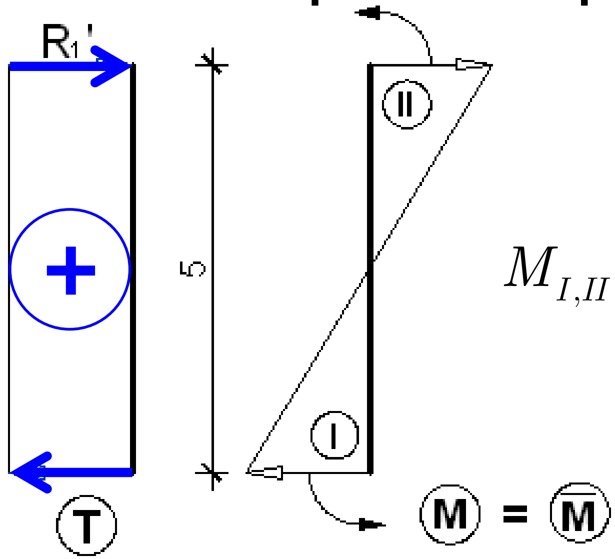
Greda: $k_{II.II'} = 12 \cdot k_{24} = 12 \cdot 31\,250 = \mathbf{375\,000\ kNm}$

2.3 Razdjelni koeficijenti na poluokviru

Čvor	Element	k_i [kNm]	Σk_i [kNm]	μ_i	$\Sigma\mu_i$
II	II-I	**22 400**	397 400	**0.056**	1.0
	II-II'	**375 000**		**0.944**	

2.4 Prijenosni koeficijent. α = –1 (2. faza proračuna po CW)

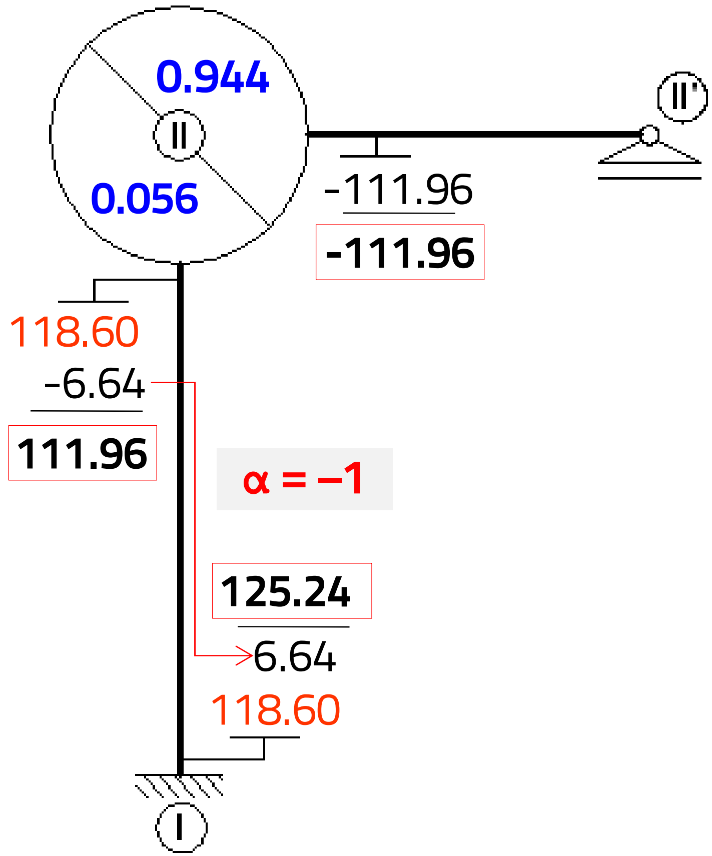
2.5 Momenti upetosti na poluokviru



$$M_{I,II} = M_{II,I} = \frac{R_1' \cdot h}{2} = \frac{+47.44 \cdot 5}{2} = +118.6 \text{ kNm}$$

2.6 Postupak iteracije na poluokviru po Csonka-Werneru

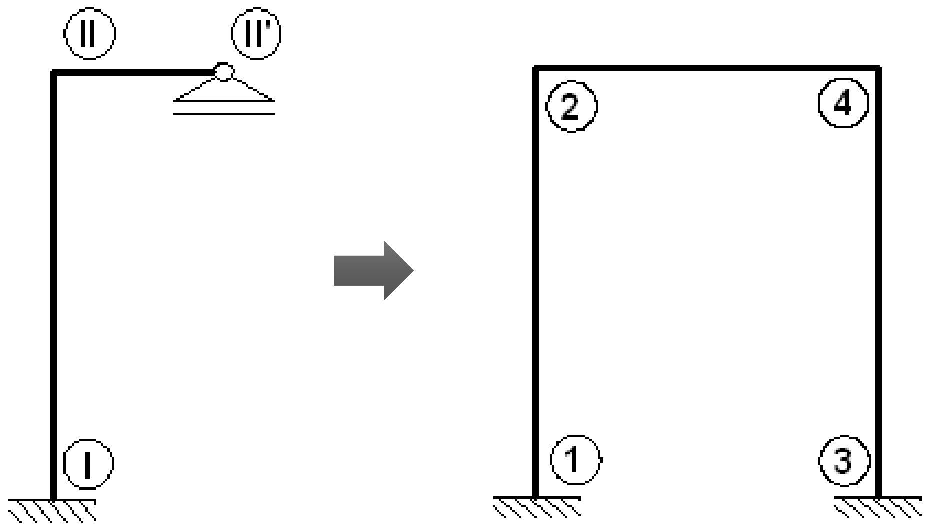
$\alpha = -1$



Koraci iteracija:

$\Rightarrow M_{II} = +\,118.60 \Rightarrow -\,118.60$ kNm

2.7 Prijenos momenata sa poluokvira na originalni sustav



Momenti na poluokviru:

Stup: $M_{I,II} = +125.24$ kNm

$M_{II,I} = +111.96$ kNm

Greda: $M_{II,II'} = -111.96$ kNm

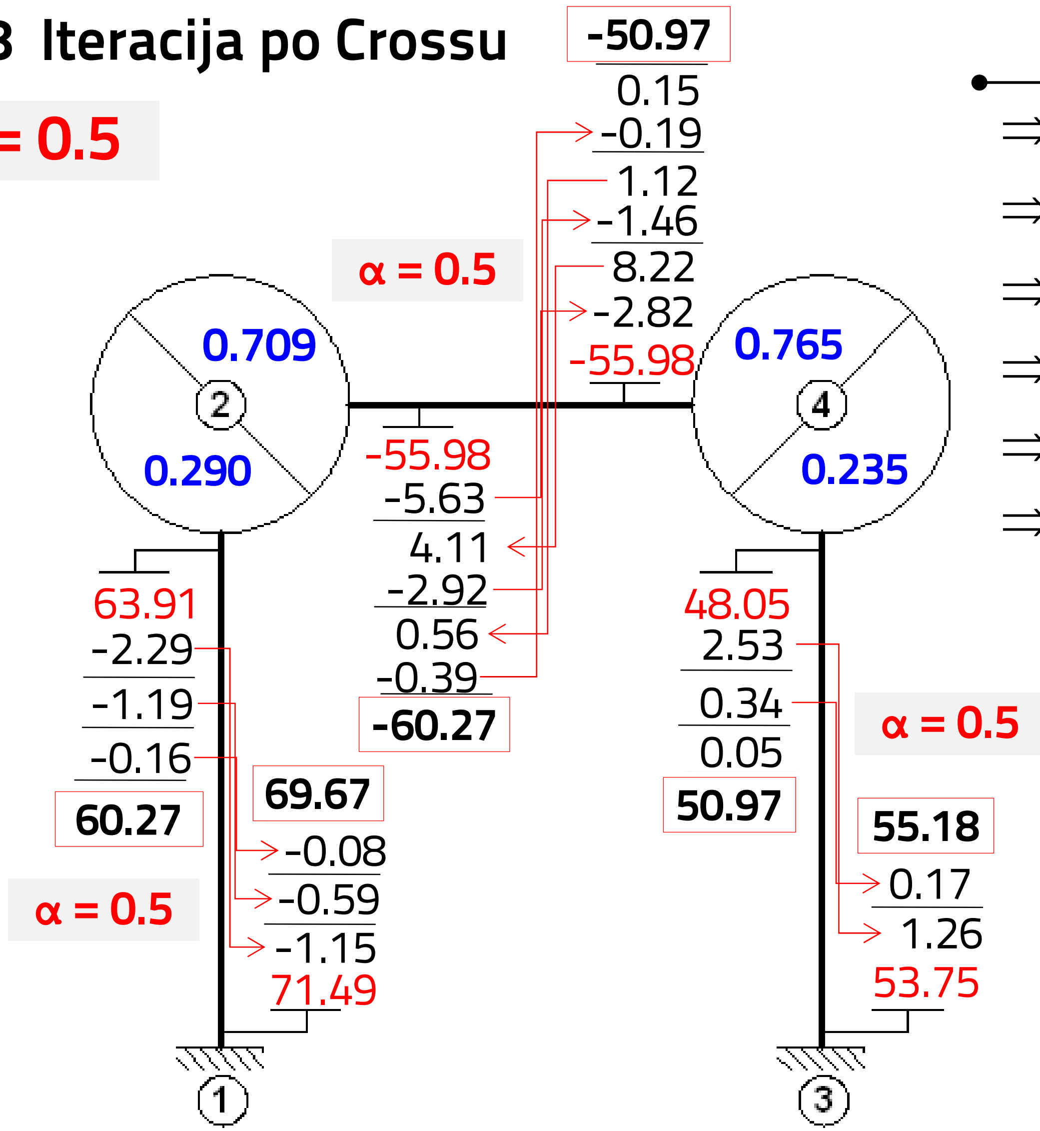
$$M_{12} = \frac{k_{12}}{k_{12}+k_{34}} \cdot M_{I,II} = +71.49 \text{ kNm} \quad M_{34} = \frac{k_{34}}{k_{12}+k_{34}} \cdot M_{I,II} = +53.75 \text{ kNm}$$

$$M_{21} = \frac{k_{12}}{k_{12}+k_{34}} \cdot M_{II,I} = +63.91 \text{ kNm} \quad M_{43} = \frac{k_{34}}{k_{12}+k_{34}} \cdot M_{II,I} = +48.05 \text{ kNm}$$

$$M_{24} = M_{42} = \frac{M_{II,II'}}{2} = -55.98 \text{ kNm}$$

2.8 Iteracija po Crossu

$\alpha = 0.5$



Koraci iteracija:

$\Rightarrow M_2 = +\,7.93 \Rightarrow -\,7.93$ kNm

$\Rightarrow M_4 = -\,10.75 \Rightarrow +\,10.75$ kNm

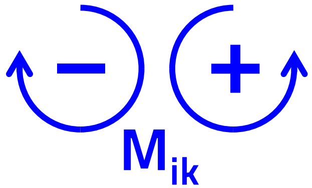
$\Rightarrow M_2 = +\,4.11 \Rightarrow -\,4.11$ kNm

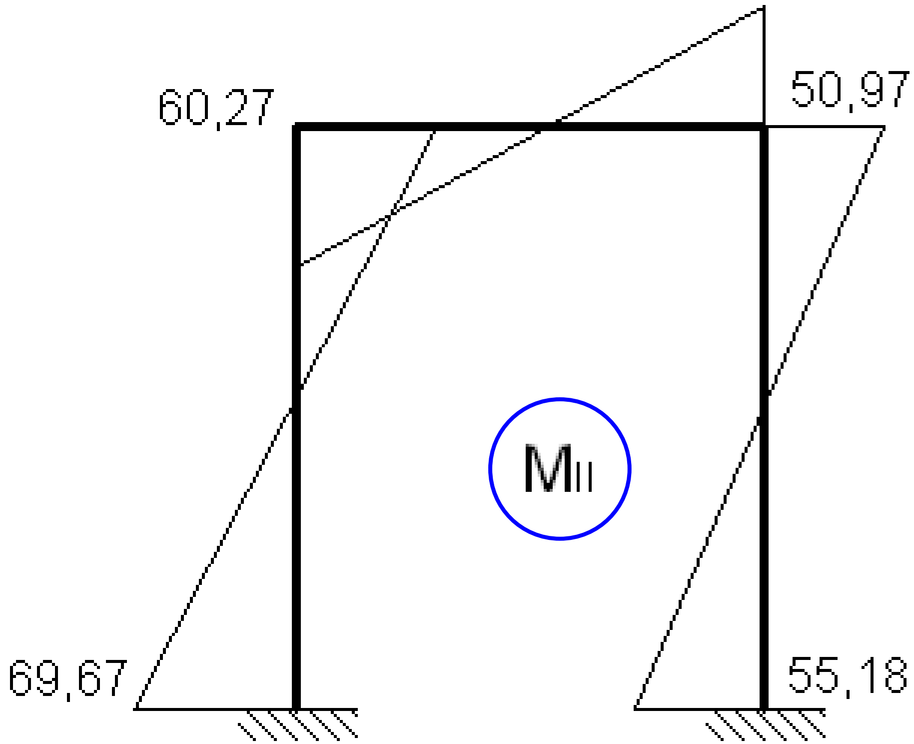
$\Rightarrow M_4 = -\,1.46 \Rightarrow +\,1.46$ kNm

$\Rightarrow M_2 = +\,0.56 \Rightarrow -\,0.56$ kNm

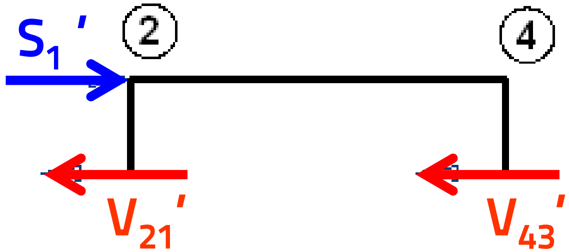
$\Rightarrow M_4 = -\,0.19 \Rightarrow +\,0.19$ kNm

2.9 M_{II} dijagram





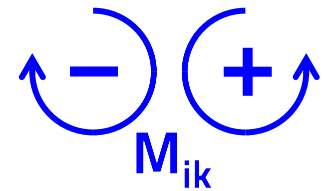
2.10 Kontrola poprečnih sila nakon provedenih iteracija



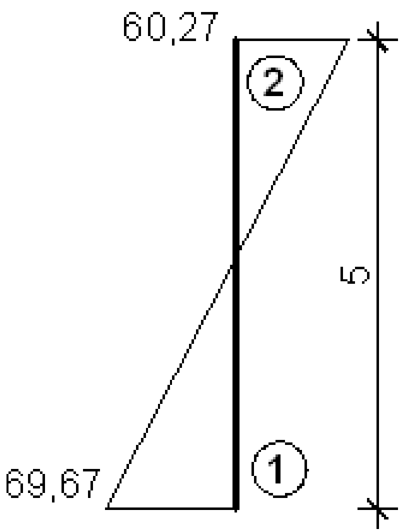
$$+S_1' = +V_{21}' + V_{43}'$$

$$+R_1 = +V_{21} + V_{43}$$

$$V_{ik} = \frac{\pm M_i \pm M_k}{L_{ik}}$$

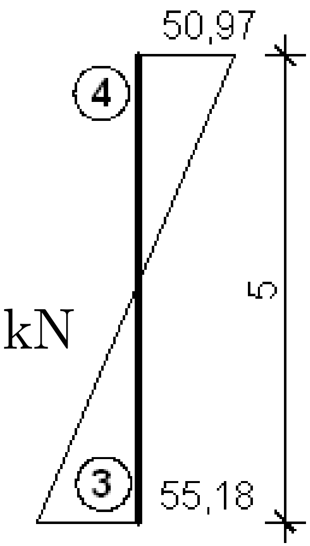


Poprečne sile iz $\mathbf{M_{II}}$ dijagrama određujemo pomoću <u>diferencijalnih odnosa!</u>

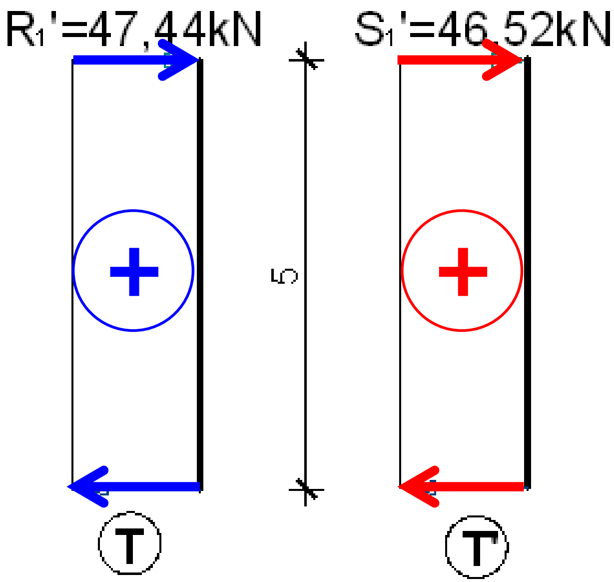


$$V_{21}' = \frac{+69.67 + 60.27}{5} = +25.99\ \mathrm{kN}$$

$$V_{43}' = \frac{+55.18 + 50.97}{5} = +21.23\ \mathrm{kN}$$



$$S_1' = V_{21}' + V_{43}' = +25.99 + 21.23 = +46.52\ \mathrm{kN}$$



$$\beta = \frac{\sum_{i=1}^{n} |R_i'| \cdot h_i}{\sum_{i=1}^{n} |S_i'| \cdot h_i}$$

Popravni koeficijent

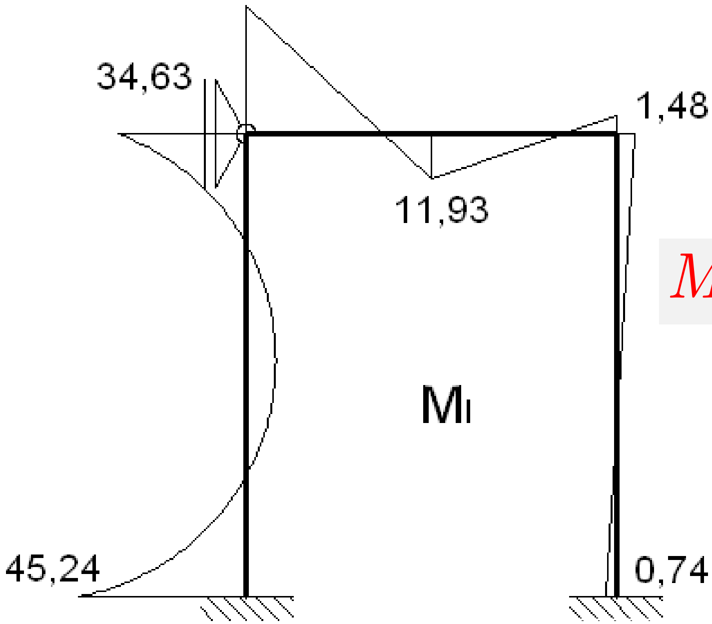
$$\beta = \frac{|R_1'| \cdot h_1}{|S_1'| \cdot h_1} = \frac{47.44 \cdot 5}{46.52 \cdot 5} = 1.0198 \approx 1.02$$

2.11 Konačni momentni dijagram (superpozicija)

$$M = M_I + \beta \cdot M_{II}$$

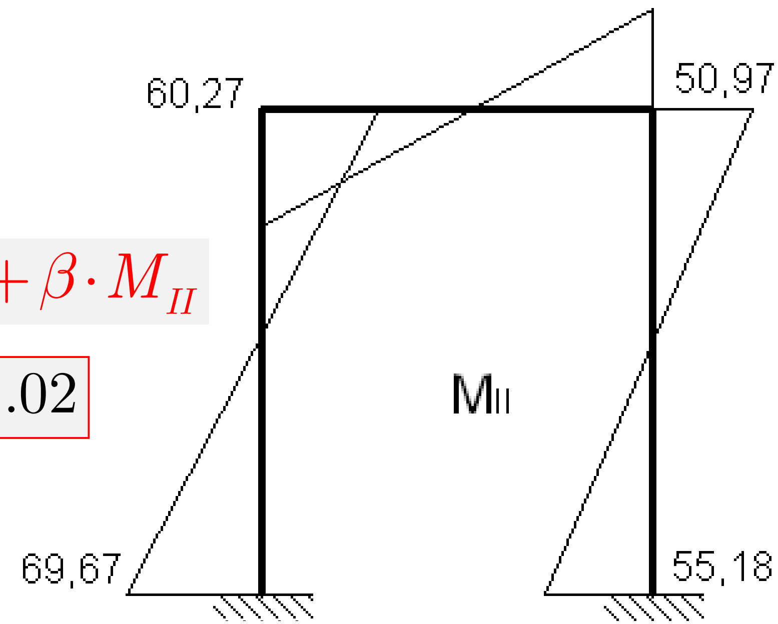
M_I - dijagram dobiven metodom Crossa na pridržanom sustavu.

M_{II} - dijagram dobiven uravnoteženjem po Crossu na pomičnom sustavu



$$M = M_I + \beta \cdot M_{II}$$

$$\beta \approx 1.02$$



$$M_{12} = 45.24 + (1.02) \cdot 69.67 = +116.30 \text{ kNm}$$

$$M_{21} = -34.63 + (1.02) \cdot 60.27 = +26.84 \text{ kNm}$$

$$M_{24} = 34.63 + (1.02) \cdot (-60.27) = -26.84 \text{ kNm}$$

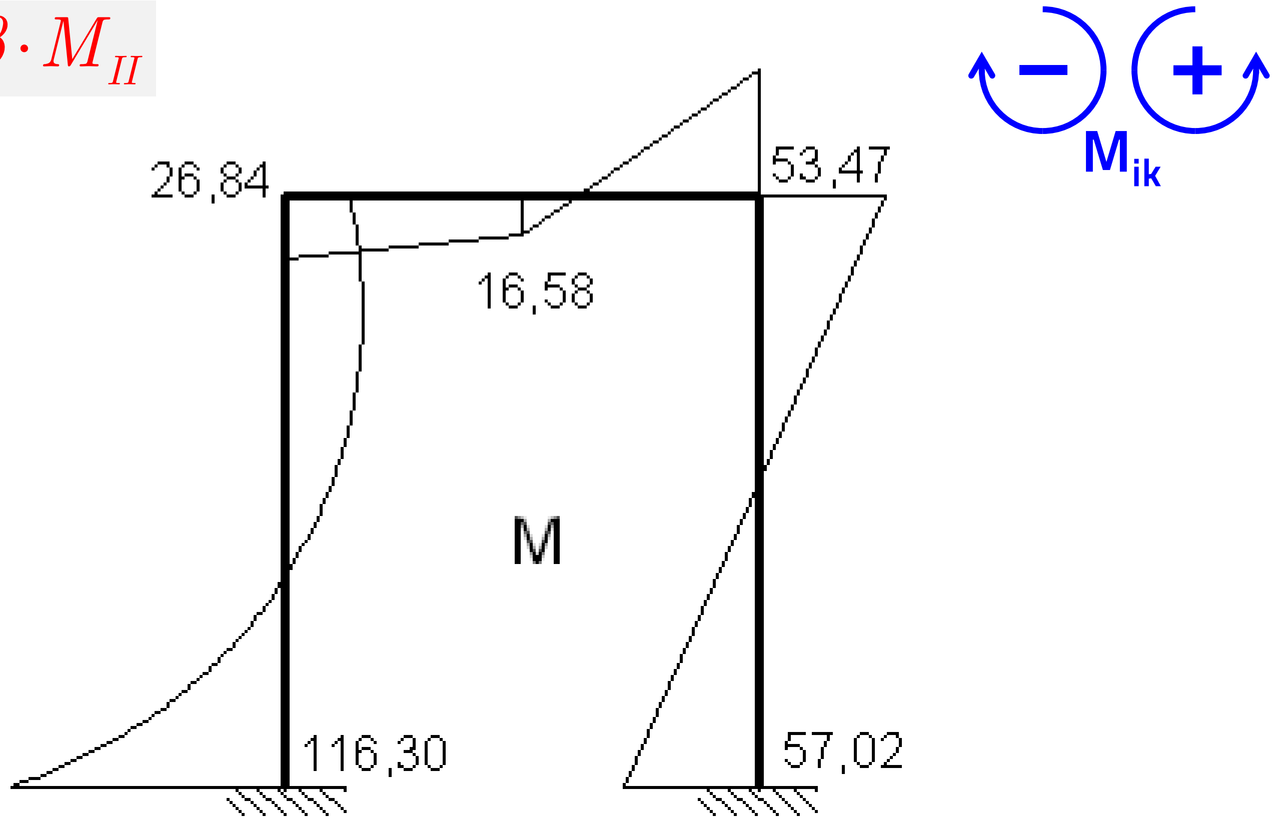
$$M_{42} = -1.48 + (1.02) \cdot (-50.97) = -53.47 \text{ kNm}$$

$$M_{43} = 1.48 + (1.02) \cdot 50.97 = +53.47 \text{ kNm}$$

$$M_{34} = 0.74 + (1.02) \cdot 55.18 = +57.02 \text{ kNm}$$

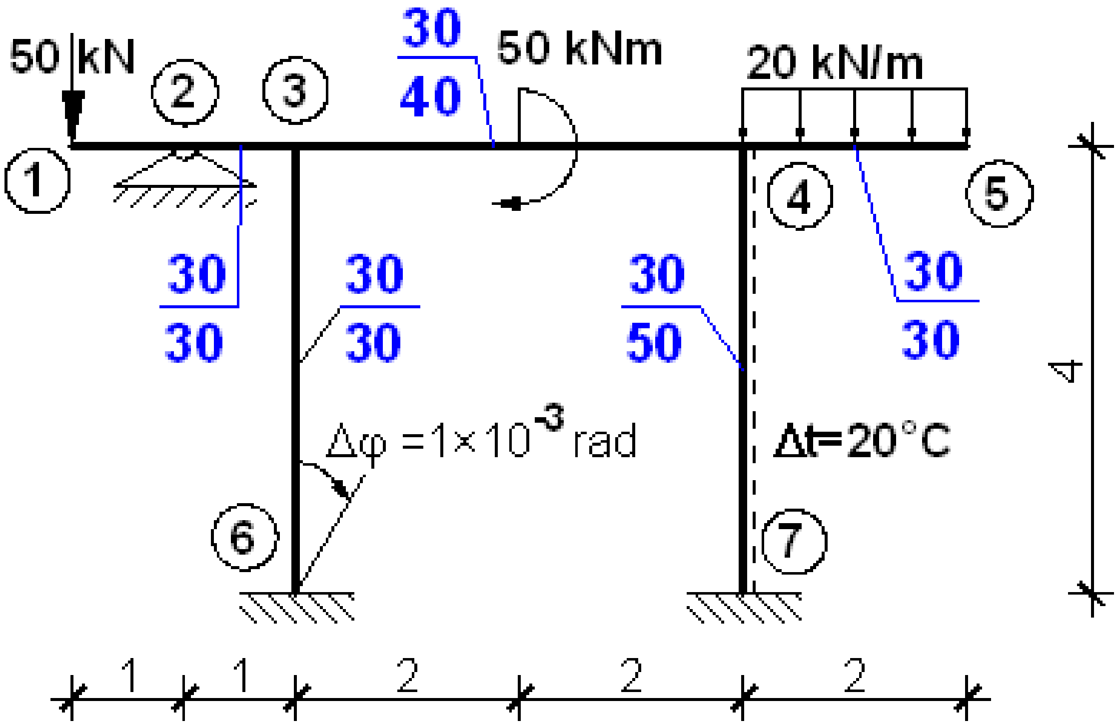
2.11 Konačni momentni dijagram (dobiven superpozicijom)

$$M = M_I + \beta \cdot M_{II}$$



Zadatak #3

Metodom Crossa odrediti dijagram momenata savijanja.



$$E = 30\ \text{GPa}$$

$$\alpha_T = \frac{1 \cdot 10^{-5}}{1°C}$$

1. Proračun krutosti

$$\text{Presjek } 30/30 \Rightarrow EI = 20\,250 \text{ kNm}^2$$

$$\text{Presjek } 30/50 \Rightarrow EI = 93\,750 \text{ kNm}^2$$

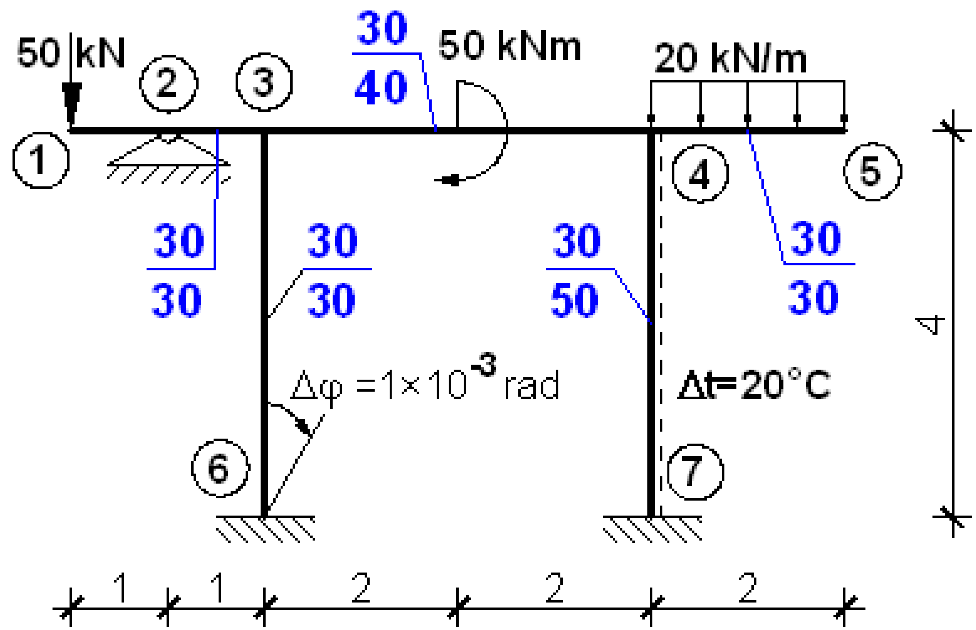
$$\text{Presjek } 30/40 \Rightarrow EI = 48\,000 \text{ kNm}^2$$

$$k_{23} = \frac{3}{4} \cdot \frac{20\,250}{1} = 15\,187.5 \text{ kNm}$$

$$k_{34} = \frac{48\,000}{4} = 12\,000 \text{ kNm}$$

$$k_{36} = \frac{20\,250}{4} = 5\,062.5 \text{ kNm}$$

$$k_{47} = \frac{93\,750}{4} = 23\,437.5 \text{ kNm}$$

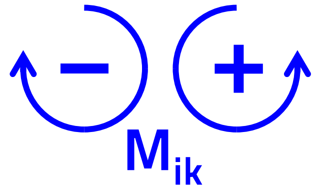


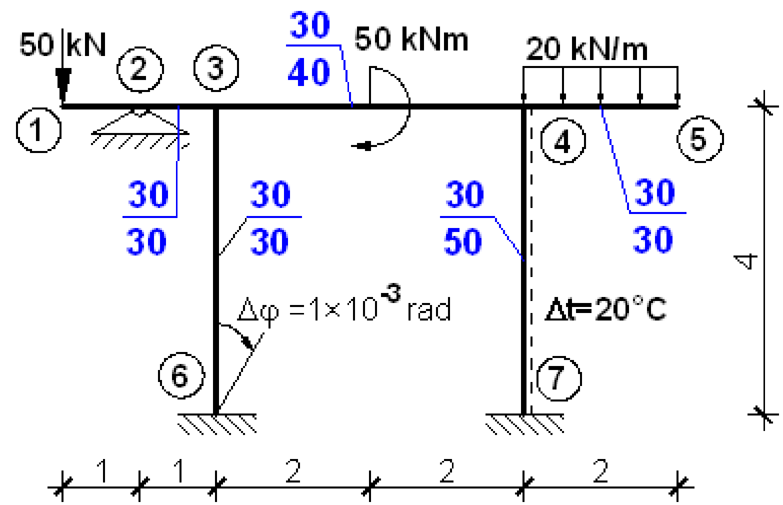
2. Razdjelni koeficijenti za iteracije po Crossu

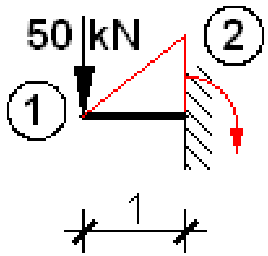
Čvor	Element	k_i [kNm]	Σk_i [kNm]	μ_i	$\Sigma \mu_i$
3	3-2	**15 187.5**	32 250	**0.471**	1.0
	3-4	**12 000**		**0.372**	
	3-6	**5 062.5**		**0.157**	
4	4-3	**12 000**	35 437.5	**0.339**	1.0
	4-7	**23 437.5**		**0.661**	

3. Prijenosni koeficijent. $\alpha = 0.5$ (1. faza proračuna po Crossu)

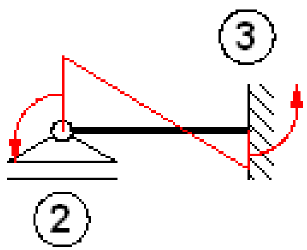
4. Momenti upetosti





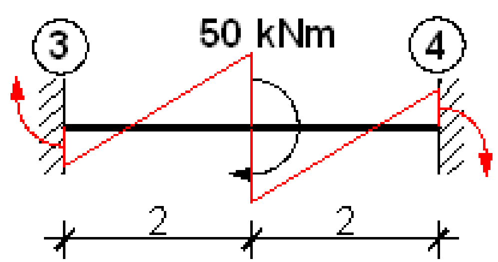


$$\bar{M}_{21} = -50.0 \text{ kNm}$$



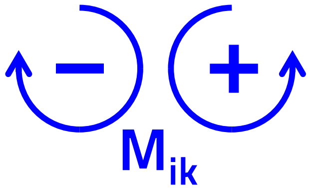
$$\bar{M}_{23} = +50.0 \text{ kNm}$$

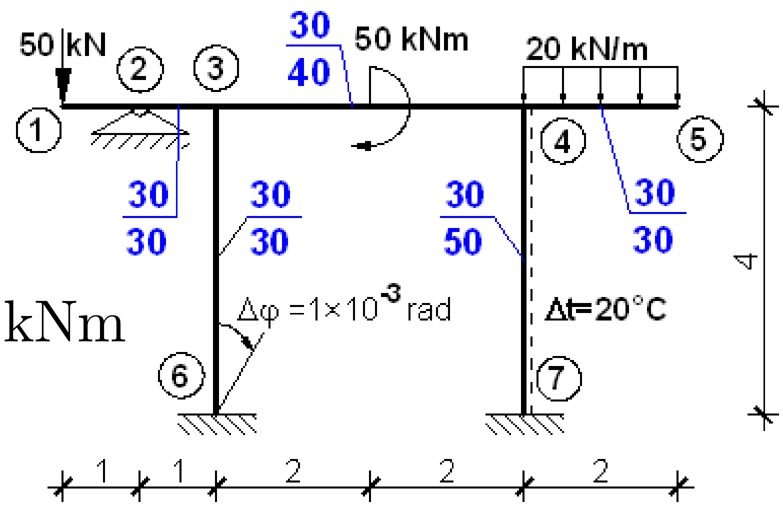
$$\bar{M}_{32} = +25.0 \text{ kNm}$$

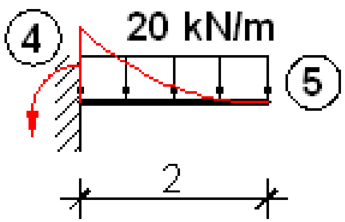


$$\bar{M}_{34} = \bar{M}_{43} = -\frac{M}{4} = -12.5 \text{ kNm}$$

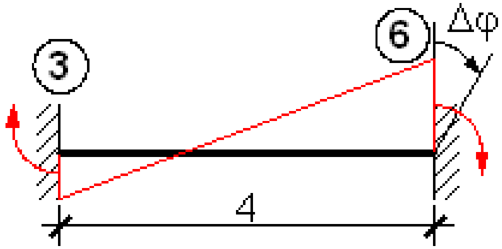
4. Momenti upetosti





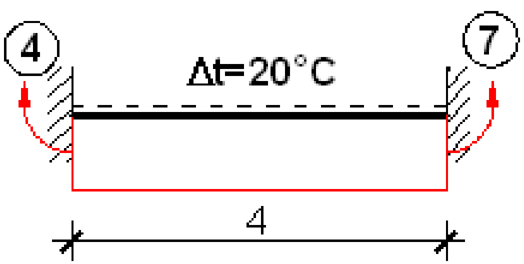


$$\bar{M}_{45} = \frac{qL^2}{2} = +40 \text{ kNm}$$



$$\bar{M}_{63} = -4 \cdot k_{36} \cdot \Delta\theta = -20.25 \text{ kNm}$$

$$\bar{M}_{36} = -2 \cdot k_{36} \cdot \Delta\theta = -10.13 \text{ kNm}$$



$$\bar{M}_{47} = -\frac{EI \cdot \alpha_T \cdot \Delta t}{h} = -37.5 \text{ kNm}$$

$$\bar{M}_{74} = +37.5 \text{ kNm}$$

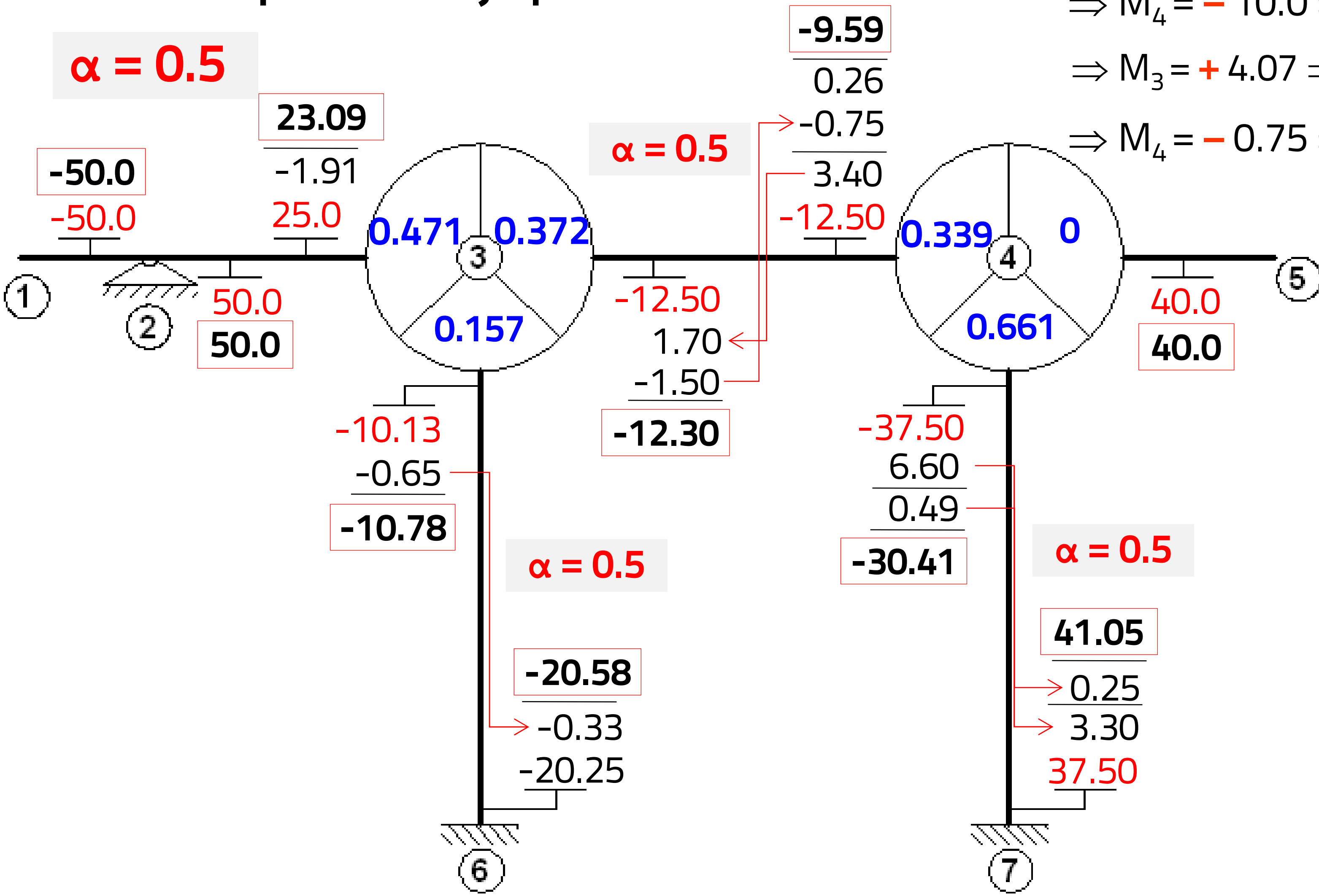
5. Postupak iteracije po Crossu

Koraci iteracija:

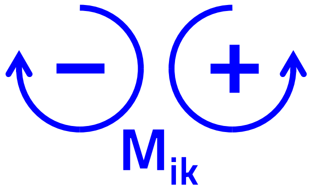
$\Rightarrow M_4 = -10.0 \Rightarrow +10.0$ kNm

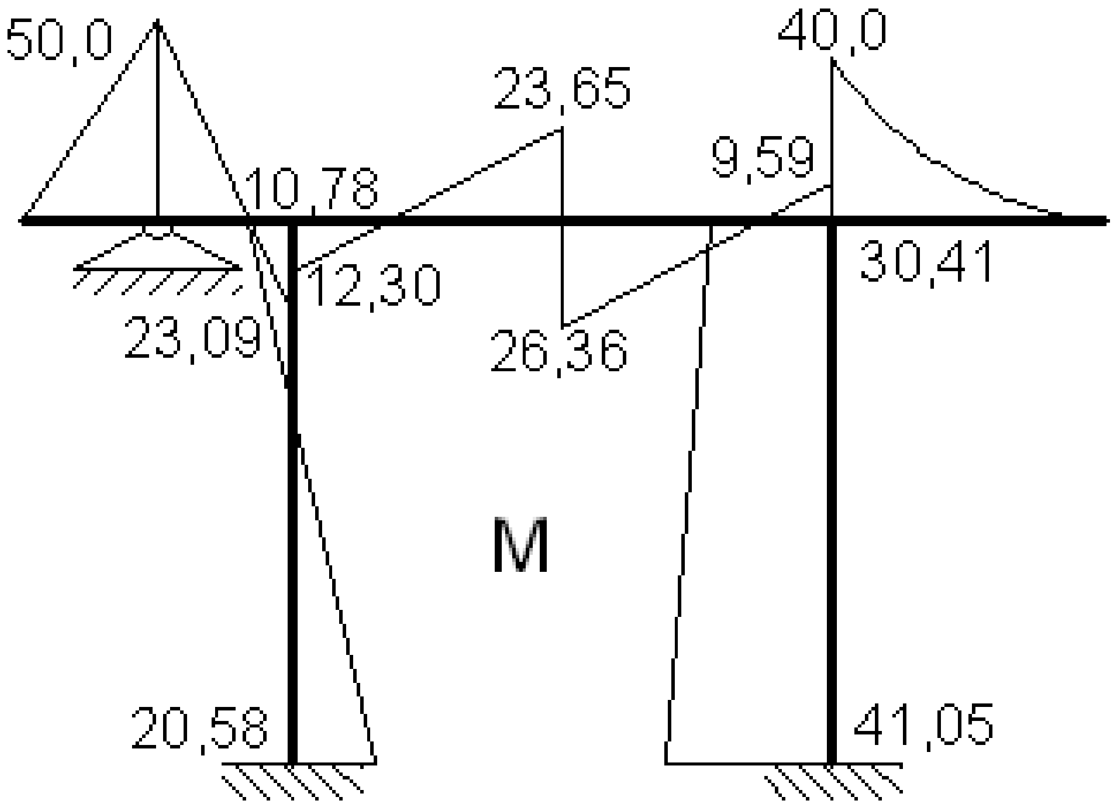
$\Rightarrow M_3 = +4.07 \Rightarrow -4.07$ kNm

$\Rightarrow M_4 = -0.75 \Rightarrow +0.75$ kNm



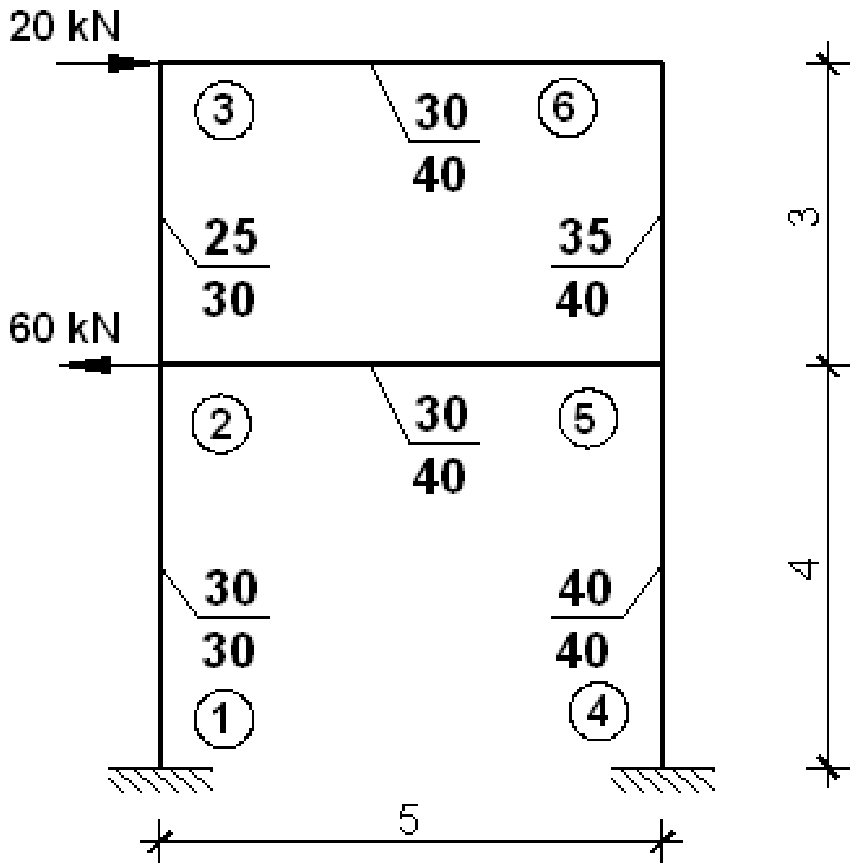
6. Konačni momentni dijagram





Zadatak #4

Kombinacijom metoda Csonka-Wernera i Crossa odrediti dijagram momenata savijanja.



$$E = 30\ \text{GPa}$$

Proračun po metodi Csonka-Wernera (2. faza proračuna)

1. Proračun krutosti

$$EI_{12} = 20\,250 \text{ kNm}^2; \qquad EI_{23} = 16\,875 \text{ kNm}^2$$

$$EI_{45} = 64\,000 \text{ kNm}^2; \qquad EI_{56} = 56\,000 \text{ kNm}^2$$

$$EI_{25} = EI_{36} = 48\,000 \text{ kNm}^2$$

$$k_{12} = \frac{20\,250}{4} = 5\,062.5 \text{ kNm}; \qquad k_{23} = \frac{16\,875}{3} = 5\,625 \text{ kNm}$$

$$k_{25} = k_{36} = \frac{48\,000}{5} = 9\,600 \text{ kNm}; \qquad k_{45} = \frac{64\,000}{4} = 16\,000 \text{ kNm}$$

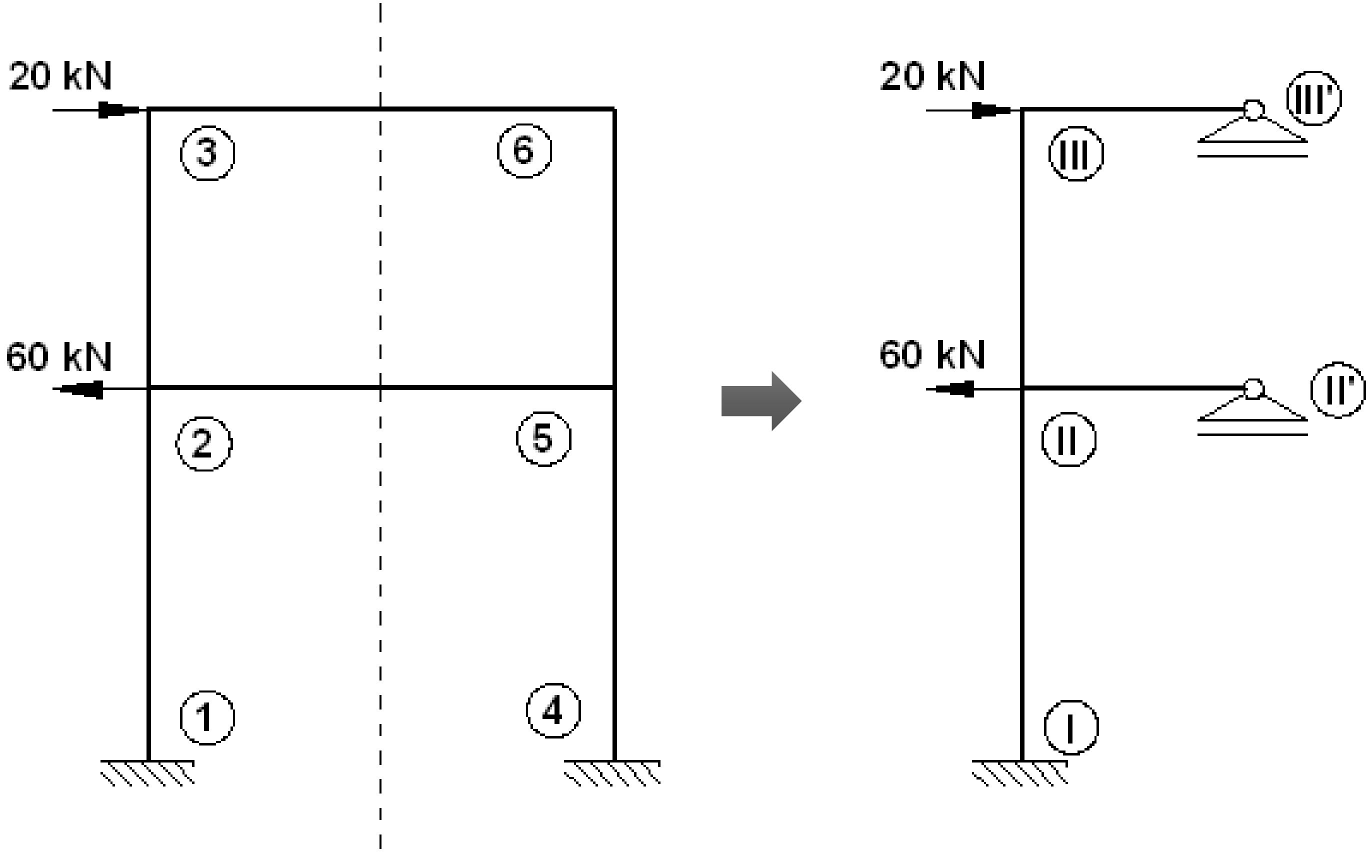
$$k_{56} = \frac{56\,000}{3} = 18\,666.66 \text{ kNm}$$

2. Razdjelni koeficijenti za iteraciju po Crossu

Čvor	Element	k_i [kNm]	Σk_i [kNm]	μ_i	$\Sigma \mu_i$
2	2-1	5 062.5	20 287.5	0.250	1.0
	2-3	5 625		0.277	
	2-5	9 600		0.473	
3	3-2	5 625	15 225	0.369	1.0
	3-6	9 600		0.631	
5	5-4	16 000	44 266.67	0.361	1.0
	5-2	9 600		0.217	
	5-6	18 666.67		0.422	
6	6-3	9 600	28 266.67	0.340	1.0
	6-5	18 666.67		0.660	

3. Prijenosni koeficijent. $\alpha = 0.5$ (1. faza proračuna po Crossu)

4. Statički sustav poluokvira



5. Proračun krutosti poluokvira

Stup: $k_{I.II} = k_{12} + k_{45} = 5\,062.5 + 16\,000 = \mathbf{22\,025.5\ kNm}$

$k_{II.III} = k_{23} + k_{56} = 5\,625 + 18\,666.67 = \mathbf{24\,291.67\ kNm}$

Greda: $k_{II.II'} = 12 \cdot k_{25} = \mathbf{115\,200\ kNm}$

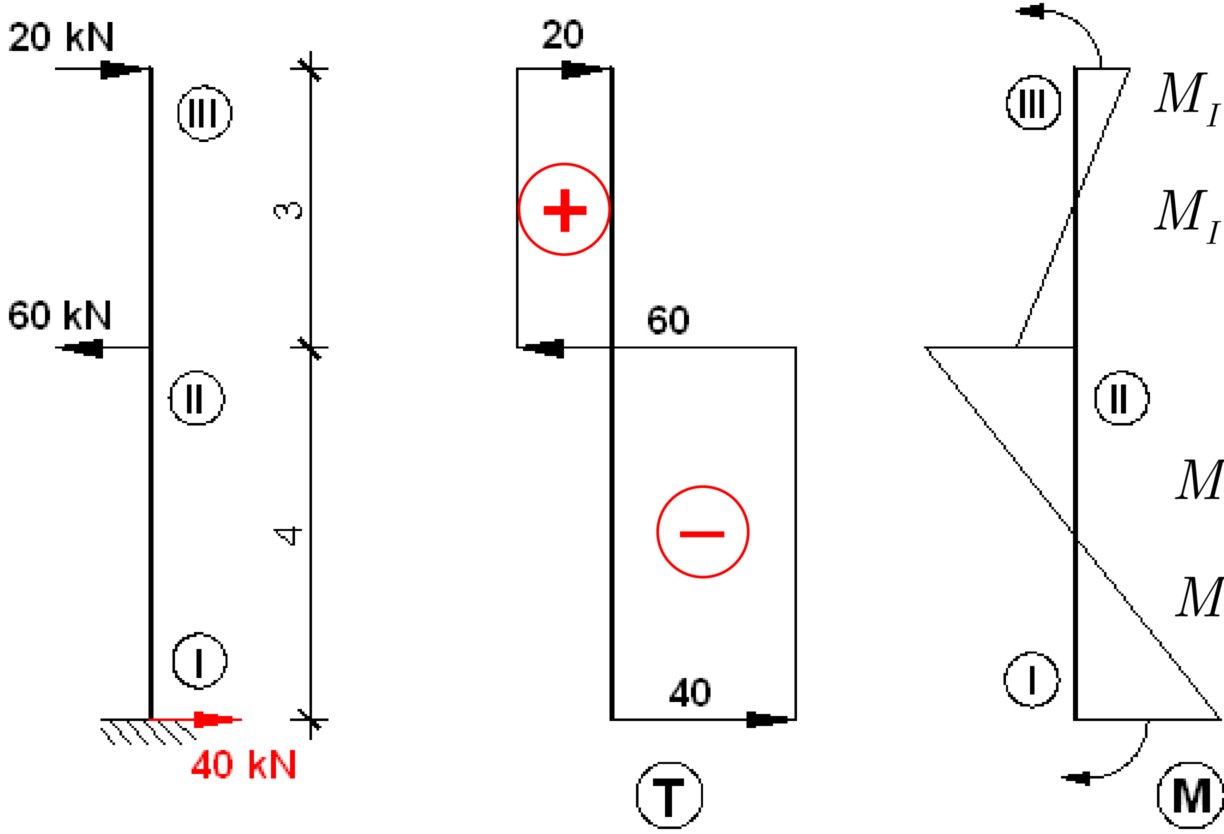
$k_{III.III'} = 12 \cdot k_{36} = \mathbf{115\,200\ kNm}$

6. Razdjelni koeficijenti za iteraciju po Csonka-Werneru

Čvor	Element	k_i [kNm]	Σk_i [kNm]	μ_i	$\Sigma\mu_i$
II	II-I	**22 025.5**	161 517.2	**0.136**	1.0
	II-II′	**115 200**		**0.713**	
	II-III	**24 291.67**		**0.150**	
III	III-II	**24 291.67**	139 491.7	**0.174**	1.0
	III-III′	**115 200**		**0.826**	

7. Prijenosni koeficijent. α = −1 (2. faza proračuna po CW)

8. Momenti upetosti poluokvira



$$M_{I,II} = M_{II,I} = \frac{20 \cdot 3}{2}$$

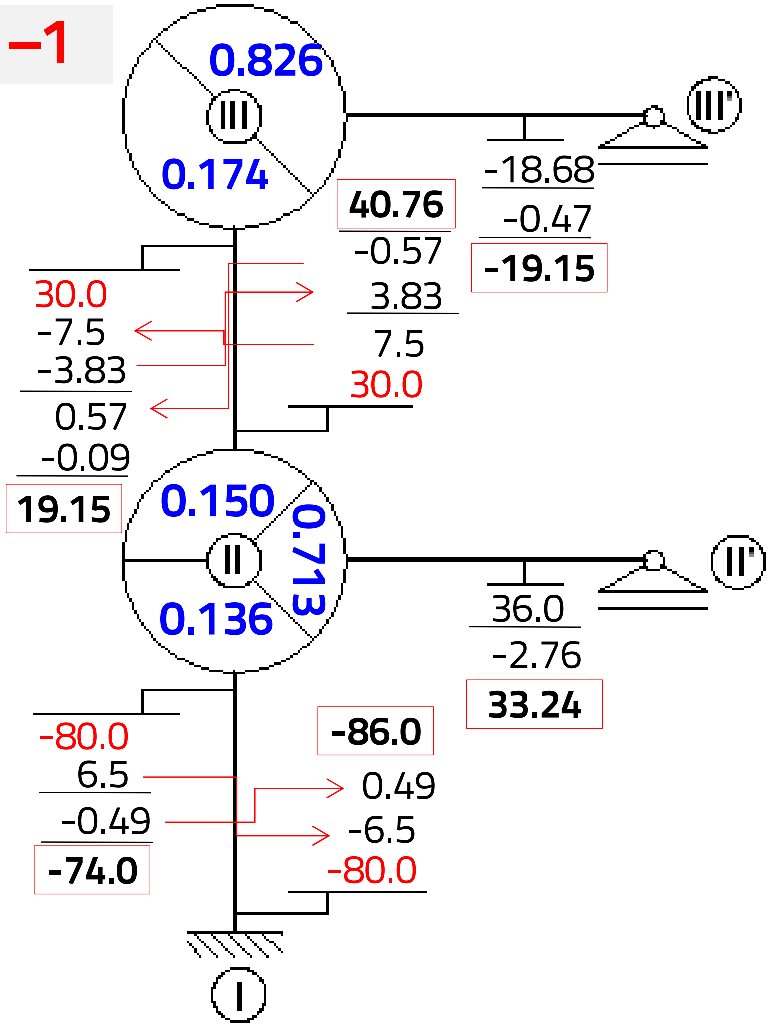
$$M_{I,II} = M_{II,I} = +30 \text{ kNm}$$

$$M_{I,II} = M_{II,I} = -\frac{40 \cdot 4}{2}$$

$$M_{I,II} = M_{II,I} = -80 \text{ kNm}$$

9. Iteracija na poluokviru po Csonka-Werneru

$\alpha = -1$



Koraci iteracija:

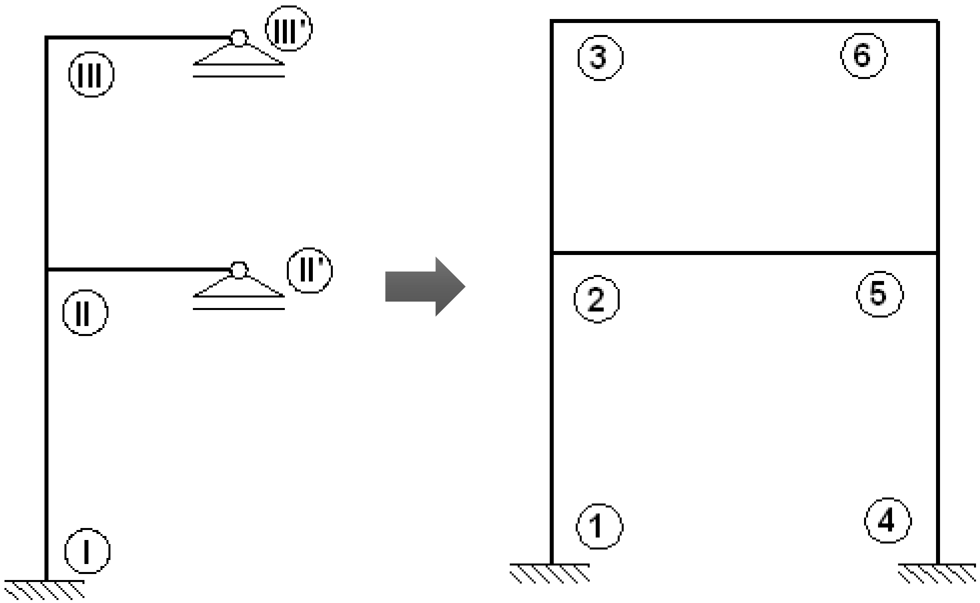
$\Rightarrow M_{II} = -50 \Rightarrow +50$ kNm

$\Rightarrow M_{III} = +30 - 7.5 = +22.5 \Rightarrow -22.5$ kNm

$\Rightarrow M_{II} = +3.83 \Rightarrow -3.83$ kNm

$\Rightarrow M_{III} = +0.57 \Rightarrow -0.57$ kNm

10. Prijenos momenata sa poluokvira na početni sustav



Momenti na poluokviru:

Stup: $M_{I.II} = -86.0$ kNm

$M_{II.I} = -74.0$ kNm

$M_{II.III} = +40.76$ kNm

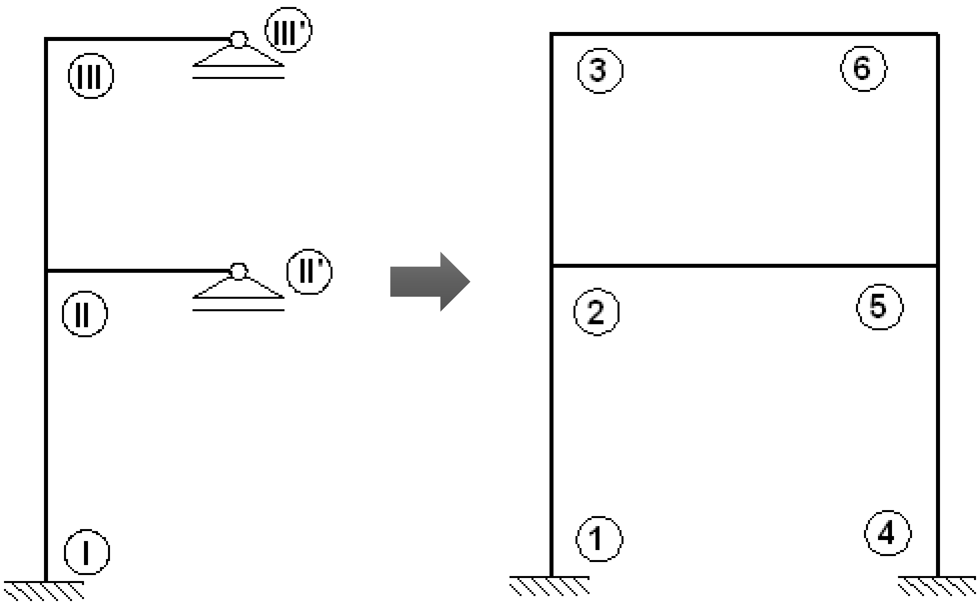
$M_{III.II} = +19.15$ kNm

Greda: $M_{II.II'} = +33.24$ kNm

$M_{III.III'} = -19.15$ kNm

$$M_{12} = \frac{k_{12}}{k_{12}+k_{45}} \cdot M_{I,II} = -20.64 \text{ kNm} \qquad M_{45} = \frac{k_{45}}{k_{12}+k_{45}} \cdot M_{I,II} = -65.36 \text{ kNm}$$

$$M_{21} = \frac{k_{12}}{k_{12}+k_{45}} \cdot M_{II,I} = -17.76 \text{ kNm} \qquad M_{54} = \frac{k_{45}}{k_{12}+k_{45}} \cdot M_{II,I} = -56.24 \text{ kNm}$$



Momenti na poluokviru:

Stup: $M_{I.II} = -86.0$ kNm

$M_{II.I} = -74.0$ kNm

$M_{II.III} = +40.76$ kNm

$M_{III.II} = +19.15$ kNm

Greda: $M_{II.II'} = +33.24$ kNm

$M_{III.III'} = -19.15$ kNm

$$M_{23} = \frac{k_{23}}{k_{23}+k_{56}} \cdot M_{II,III} = +9.34 \text{ kNm} \qquad M_{56} = \frac{k_{56}}{k_{23}+k_{56}} \cdot M_{II,III} = +31.42 \text{ kNm}$$

$$M_{32} = \frac{k_{23}}{k_{23}+k_{56}} \cdot M_{III,II} = +4.39 \text{ kNm} \qquad M_{65} = \frac{k_{56}}{k_{23}+k_{56}} \cdot M_{III,II} = +14.76 \text{ kNm}$$

$$M_{25} = M_{52} = \frac{M_{II,II'}}{2} = +16.62 \text{ kNm} \qquad M_{36} = M_{63} = \frac{M_{III,III'}}{2} = -9.58 \text{ kNm}$$

11. Iteracija po Crossu

$\alpha = 0.5$

Koraci iteracija:

$\Rightarrow M_2 = +8.20 \Rightarrow -8.20$ kNm

$\Rightarrow M_5 = -10.13 \Rightarrow +10.13$ kNm

$\Rightarrow M_6 = +7.31 \Rightarrow -7.31$ kNm

$\Rightarrow M_3 = -7.58 \Rightarrow +7.58$ kNm

$\Rightarrow M_2 = +2.51 \Rightarrow -2.51$ kNm

$\Rightarrow M_5 = -3.0 \Rightarrow +3.0$ kNm

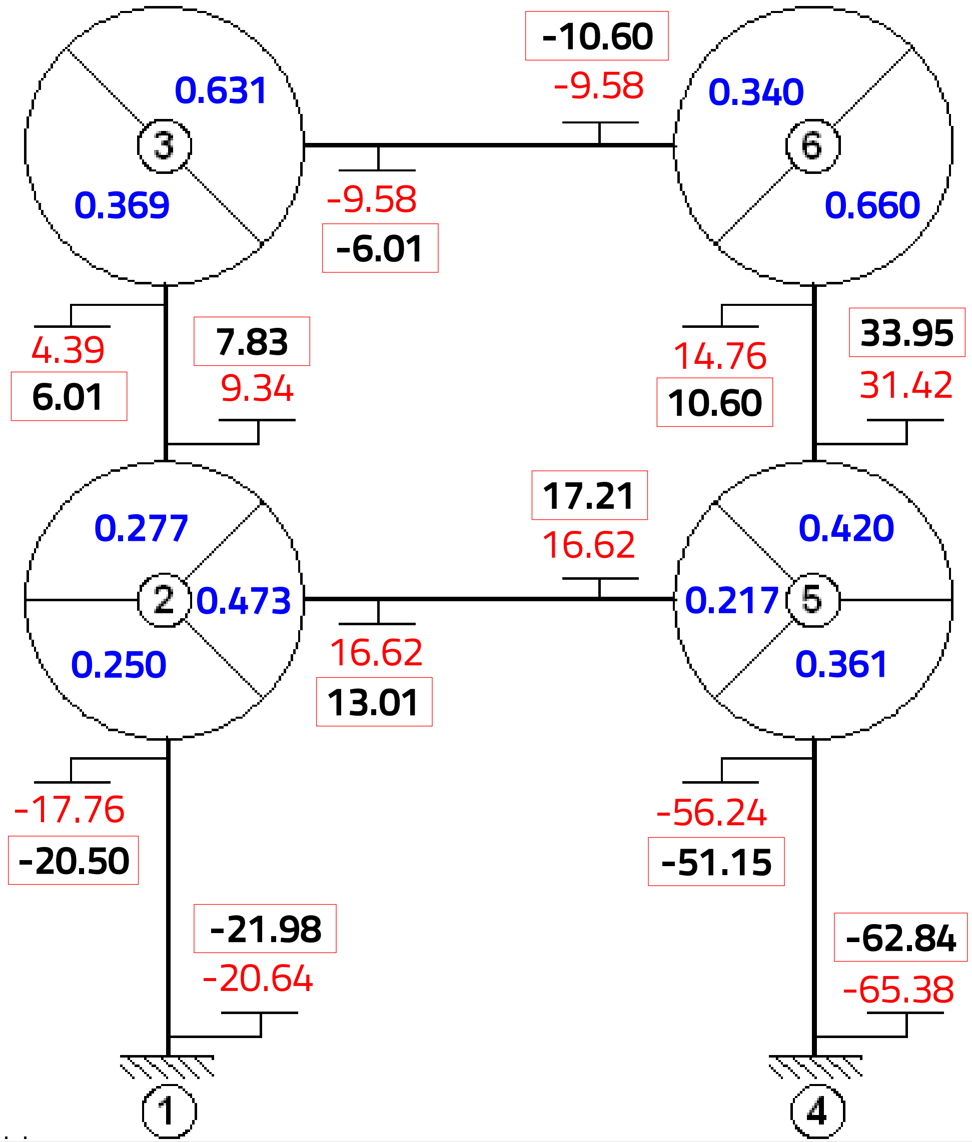
$\Rightarrow M_6 = +3.02 \Rightarrow -3.02$ kNm

$\Rightarrow M_5 = -0.99 \Rightarrow +0.99$ kNm

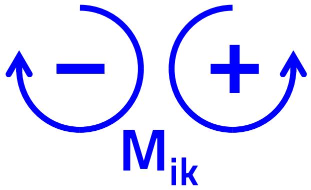
$\Rightarrow M_3 = -0.86 \Rightarrow +0.86$ kNm

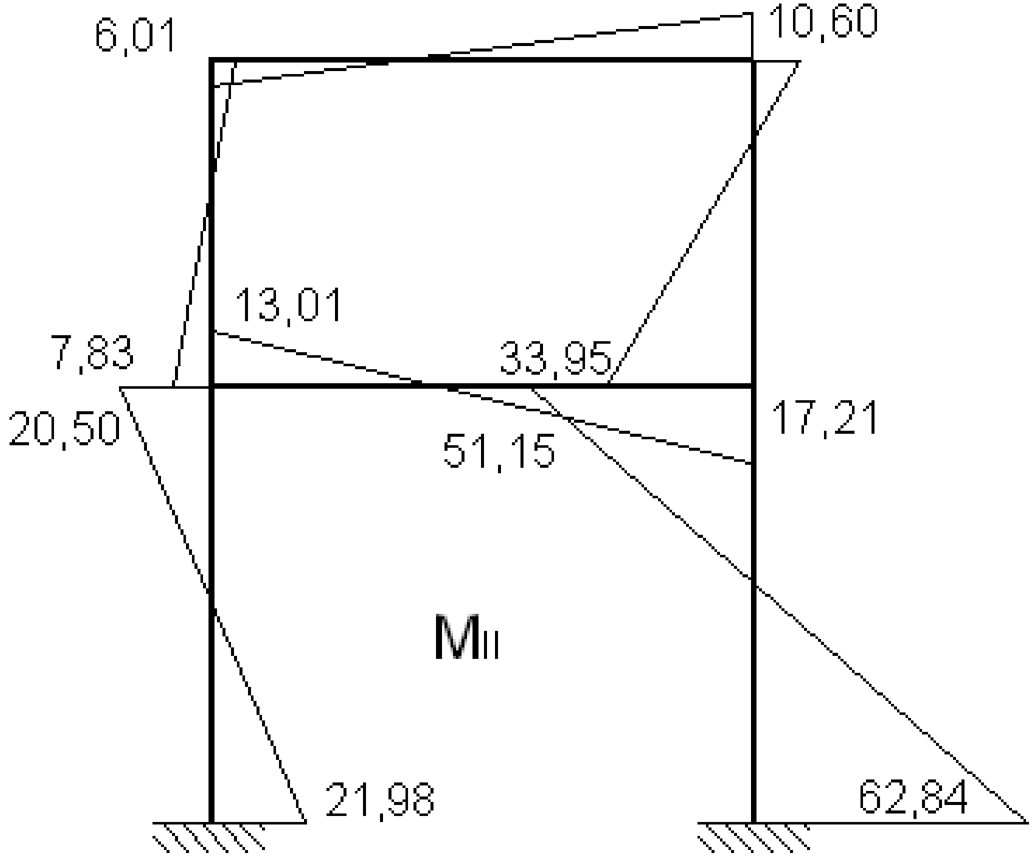
$\Rightarrow M_6 = +0.48 \Rightarrow -0.48$ kNm

$\Rightarrow M_2 = +0.27 \Rightarrow -0.27$ kNm

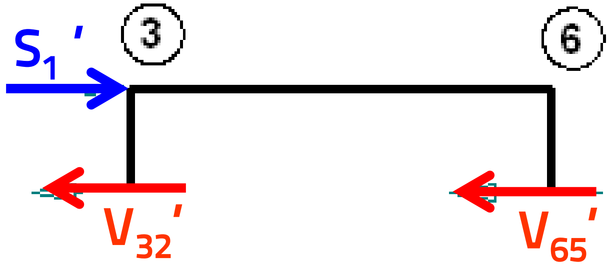


12. M_{II} dijagram



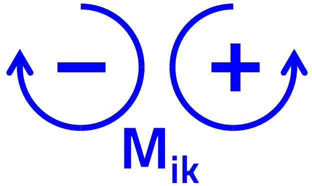


13. Kontrola poprečnih sila



$$+S_1' = +V_{32}' + V_{65}'$$

$$V_{ik} = \frac{\pm M_i \pm M_k}{L_{ik}}$$

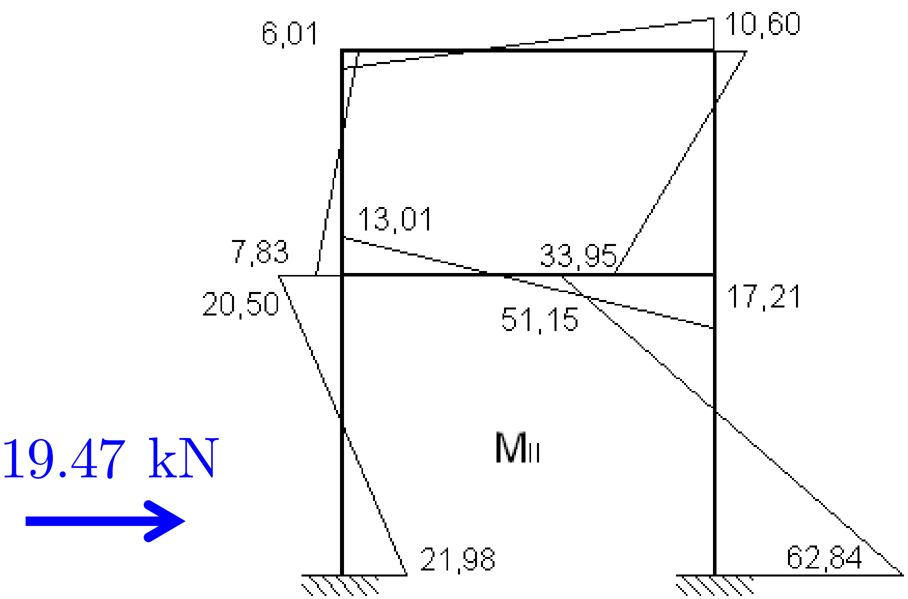


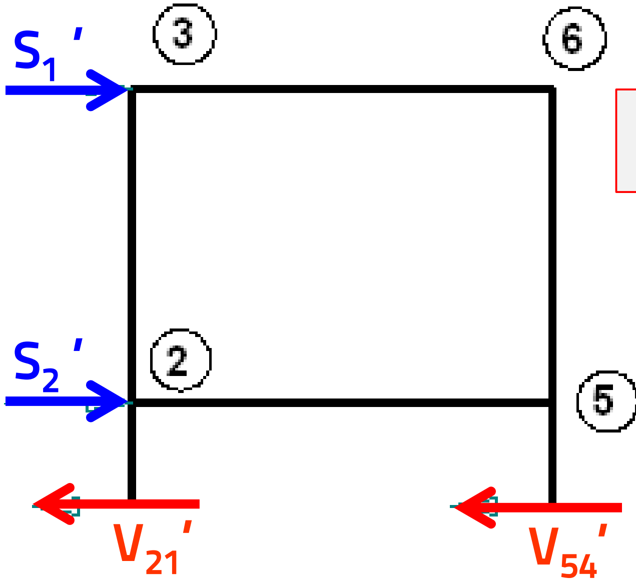
Poprečne sile iz $\mathbf{M_{II}}$ dijagrama određujemo pomoću <u>diferencijalnih odnosa!</u>

$$V_{32}' = \frac{+6.01 + 7.85}{3} = +4.62 \text{ kN}$$

$$V_{65}' = \frac{+10.60 + 33.95}{3} = +14.85 \text{ kN}$$

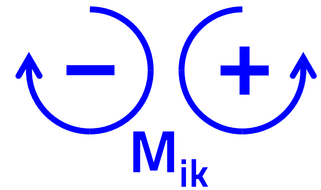
$$S_1' = V_{32}' + V_{65}' = +4.62 + 14.85 = +19.47 \text{ kN}$$





$$+S_2' = +V_{21}' + V_{54}' - S_2'$$

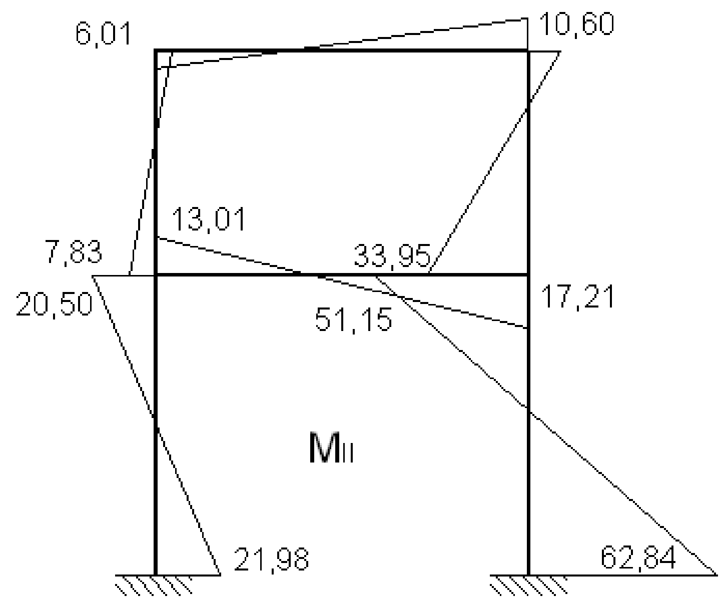
$$V_{ik} = \frac{\pm M_i \pm M_k}{L_{ik}}$$



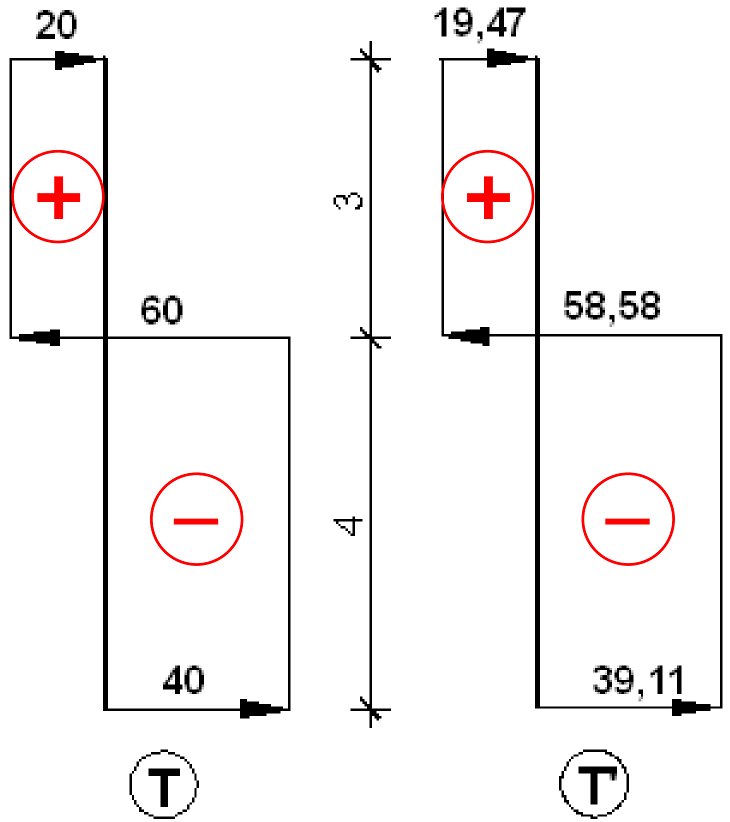
$$V_{21}' = \frac{-20.50 - 21.98}{4} = -10.62 \text{ kN}$$

$$V_{54}' = \frac{-51.15 - 62.84}{4} = -28.49 \text{ kN}$$

$$S_2' = -10.62 - 28.49 \underbrace{-19.47}_{S_1'} = -58.58 \text{ kN}$$



14. **Popravni koeficijent** (kontrola poprečnih sila)



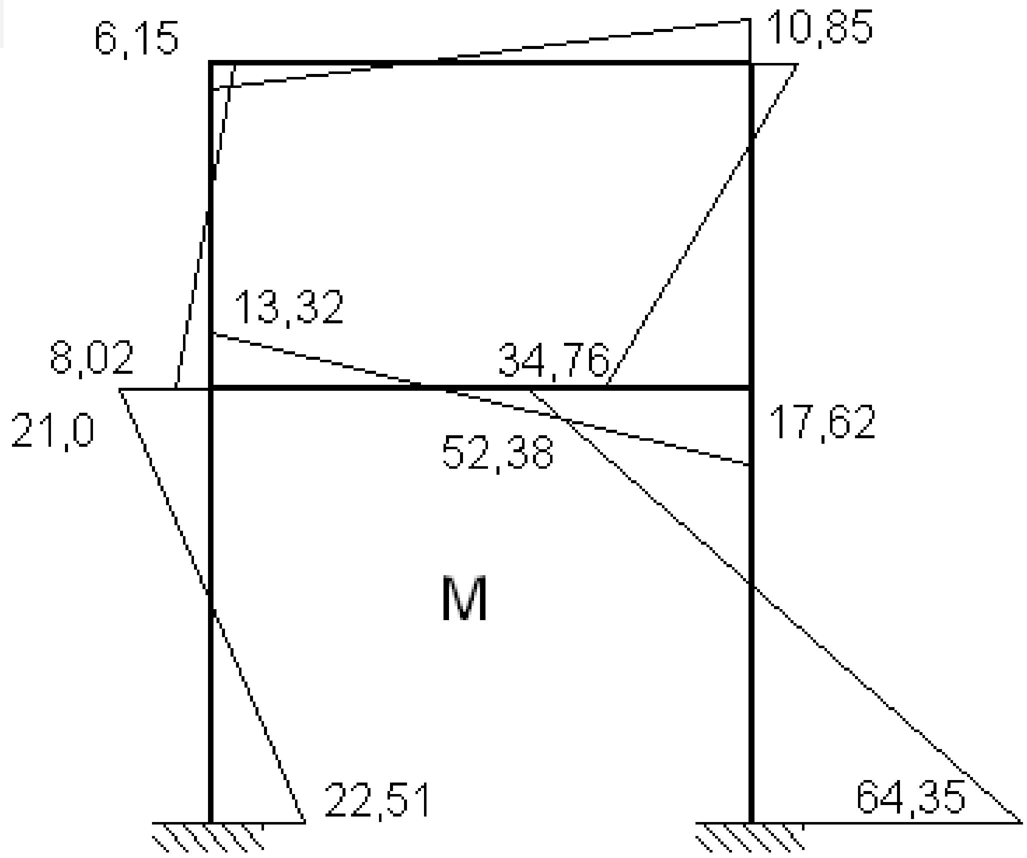
$$\beta = \frac{\sum_{i=1}^{n} |R_i'| \cdot h_i}{\sum_{i=1}^{n} |S_i'| \cdot h_i}$$

$$\beta = \frac{|R_1'| \cdot h_1 + |R_2'| \cdot h_2}{|S_1'| \cdot h_1 + |S_2'| \cdot h_2}$$

$$\beta = \frac{20 \cdot 3 + 60 \cdot 4}{19.47 \cdot 3 + 58.58 \cdot 4} = 1.0239 \approx 1.02$$

15. Konačni momentni dijagram

$$M = \beta \cdot M_{II}$$



Hvala na pažnji !
Pitanja?

Doc. dr. sc. Marin Grubišić, mag. ing. aedif.

Sveučilište u Osijeku (UNIOS)
Građevinski i arhitektonski fakultet Osijek (GrAFOS)
Zavod za tehničku mehaniku (ZTM)
Katedra/Laboratorij za eksperimentalnu mehaniku
Vladimira Preloga 3, **Ured II.26**, HR–31 000 Osijek, Hrvatska

marin.grubisic@gfos.hr

Konzultacije: **srijedom 8:00 — 9:00 sati**
Google Classroom: **qmvjpo6**