SVEUČILIŠTE JOSIPA JURJA STROSSMAYERA U OSIJEKU
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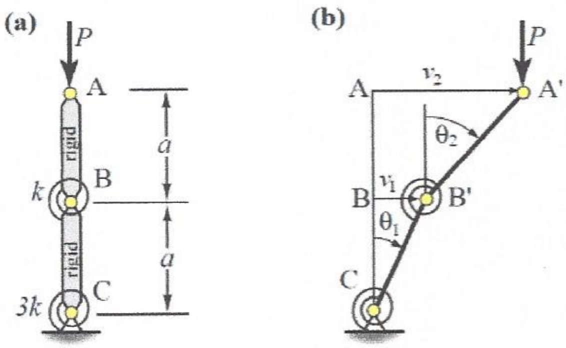
Zavod za tehničku mehaniku

STABILNOST KONSTRUKCIJA
Kolokvij

1) Dva kruta štapa jednakih duljina međusobno su zglobno spojena u čvoru B te zglobno oslonjena u čvoru C. Pri tome su slobodne rotacije u čvorovima B i C djelomično ograničene djelovanjem elastičnih rotacijskih opruga krutosti $\boldsymbol{k}$ i $\boldsymbol{3k}$, kao što je to prikazano na slici (a). Konstantna uzdužna tlačna sila $\boldsymbol{P}$ djeluje u čvoru A.

Odredite kritičnu vrijednost sile $\boldsymbol{P}$ potrebne za održavanje ravnoteže u pomaknutom položaju, principom statičke ravnoteže



2) Izvedite izraz za kritičnu silu konstrukcijskog sustava iz prethodnog primjera, ali ovaj put energetskim pristupom.

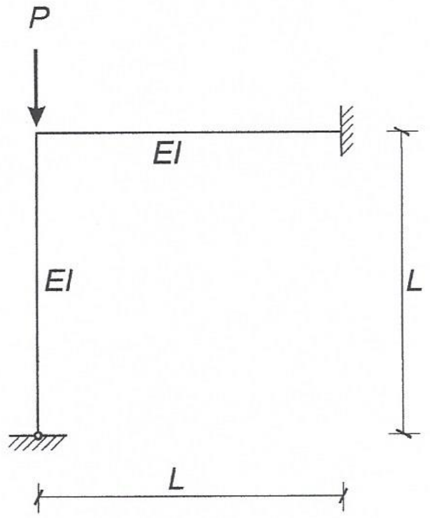
Usporedite rezultate.

3) Odredite veličinu kritične sile P_{cr} poluokvira na slici pomoću rješenja diferencijalne jednadžbe ravnoteže IV. reda

Za pretpostavljene vrijednosti:

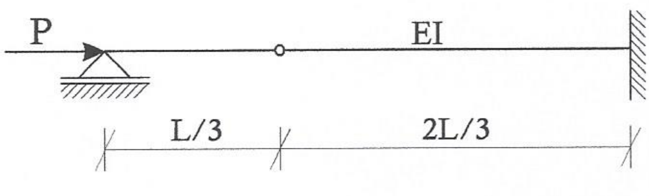
$$EI = 13600\ kNm^2 \quad \text{i} \quad L = 3{,}2\ m,$$

izračunajte veličinu kritične sile.



4) Odredite kritičnu silu poluokvira iz prethodnog zadatka metodom krutosti.

5) Odredite kritičnu silu štapa s unutarnjim zglobom prikazanim na slici..



U Osijeku, 2024. godine

RJEŠENJE DIFERENCIJALNE JEDNADŽBE RAVNOTEŽE 4. REDA S PRIPADAJUĆIM DERIVACIJAMA I VEZE S UNUTARNJIM SILAMA

Progib: $y(x) = A\sin \alpha x + B\cos \alpha x + Cx + D$

Kut zaokreta: $y'(x) = \alpha(A\cos \alpha x - B\sin \alpha x) + C$

$y''(x) = -\alpha^2 A\sin \alpha x - \alpha^2 B\cos \alpha x$;

$y'''(x) = -\alpha^3 A\cos \alpha x + \alpha^3 B\sin \alpha x$

Jednadžba momenata: $M(x) = -EIy''(x)$

Jednadžba posmika: $T(x) = -EIy'''(x) - Py'(x) = -EI\left(y''' + \alpha^2 y'\right)$

JEDNADŽBE METODE POČETNIH PARAMETARA

$$y(x) = y_0 + \frac{y_0'}{\alpha}\sin \alpha x - \frac{M_0}{\alpha^2 EI}(1 - \cos \alpha x) - \frac{T_0}{\alpha^3 EI}(\alpha x - \sin \alpha x)$$

$$y'(x) = y_0'\cos \alpha x - \frac{M_0}{\alpha EI}\sin \alpha x - \frac{T_0}{\alpha^2 EI}(1 - \cos \alpha x)$$

$$M(x) = -EIy''(x) = \alpha EIy_0'\sin \alpha x + M_0\cos \alpha x + \frac{T_0}{\alpha}\sin \alpha x$$

$$T(x) = -EI\left(y''' + \alpha^2 y'\right) = T_0$$

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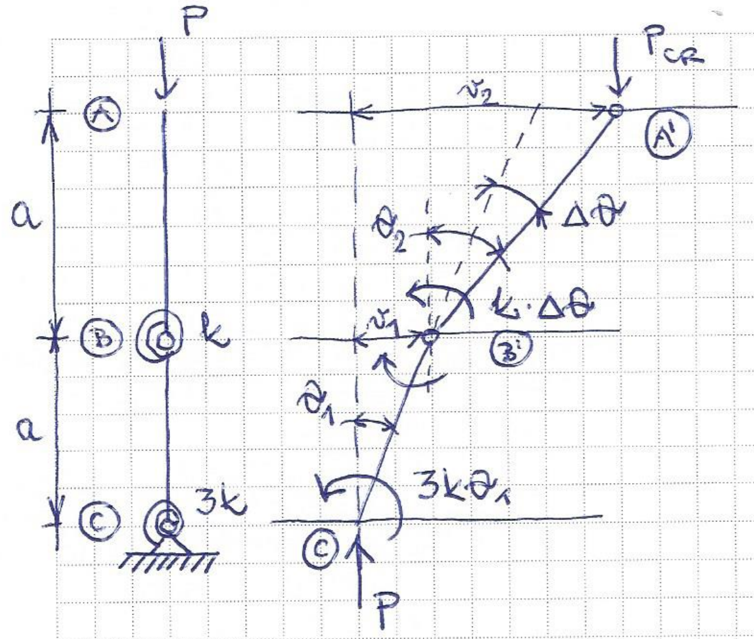

$$\vartheta_1 = \frac{v_1}{a}, \quad \vartheta_2 = \frac{v_2 - v_1}{a}$$

$$\Delta\vartheta = \vartheta_2 - \vartheta_1 = \frac{v_2 - v_1}{a} - \frac{v_1}{a} = $$

$$= \frac{v_2 - 2v_1}{a}$$

1. Statički pristup

$$+\circlearrowright \quad \Sigma M_{A'} = 3k\vartheta_1 - P_{cr}\cdot v_2 = 3k\frac{v_1}{a} - P_{cr}\cdot v_2 = 0 \quad / \cdot a$$

$$\Sigma M_{B'} = 3k\vartheta_1 - k\cdot\Delta\vartheta - P_{cr}\, v_1 =$$

$$= 3k\frac{v_1}{a} - k\cdot\frac{v_2 - 2v_1}{a} - P_{cr}\cdot v_1 = 0 \quad / \cdot a$$

$$3kv_1 - aP_{cr}v_2 = 0$$

$$3kv_1 - kv_2 + 2kv_1 - aP_{cr}v_1 = 0$$

$$\begin{bmatrix} 3k & -aP_{cr} \\ 3k+2k-aP_{cr} & -k \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det[\;] = 0$$

$$-3k^2 + 5akP_{cr} - a^2P_{cr}^2 = 0 \quad / \cdot (-1)$$

$$a^2P_{cr}^2 - 5akP_{cr} + 3k^2 = 0$$

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$$P_{cr_{1,2}} = \frac{5ak \pm \sqrt{25a^2k^2 - 12a^2k^2}}{2a^2} = \frac{5ak \pm \sqrt{13}\, ak}{2a^2}$$

$$= (5 \pm \sqrt{13}) \frac{k}{2a}$$

$$P_{cr} = 0{,}697\ k/a$$

(2) Energetski pristup

– ukupni vertikalni pomak hvatišta sile P (točka A) uslijed zaokreta štapova:

$$\Delta = \Delta_1 + \Delta_2 = \frac{a}{2}\vartheta_1^2 + \frac{a}{2}\vartheta_2^2 = \frac{a}{2}\cdot\frac{v_1^2}{a^2} + \frac{a}{2}\frac{(v_2-v_1)^2}{a^2} =$$

$$= \frac{v_1^2}{2a} + \frac{v_2^2 - 2v_1v_2 + v_1^2}{2a} = \frac{2v_1^2 - 2v_1v_2 + v_2^2}{2a}$$

$$V = -P\Delta = -P_{cr}\frac{2v_1^2 - 2v_1v_2 + v_2^2}{2a}$$

$$U = \frac{1}{2}3k\vartheta_1^2 + \frac{1}{2}k(\Delta\vartheta)^2 = \frac{1}{2}3k\frac{v_1^2}{a^2} + \frac{1}{2}k\frac{v_2^2 - 4v_1v_2 + 4v_1^2}{a^2} =$$

$$= \frac{7kv_1^2 + kv_2^2 - 4kv_1v_2}{2a^2}$$

$$\Pi = U + V = \frac{7kv_1^2 + kv_2^2 - 4kv_1v_2}{2a^2} - P_{cr}\frac{2v_1^2 - 2v_1v_2 + v_2^2}{2a}$$

$$\frac{\partial \Pi}{\partial v_1} = \frac{1}{2a^2}(14kv_1 - 4kv_2) - \frac{P_{cr}}{2a}(4v_1 - 2v_2) = \emptyset \quad / \cdot 2a^2$$

$$\frac{\partial \Pi}{\partial v_2} = \frac{1}{2a^2}(2kv_2 - 4kv_1) - \frac{P_{cr}}{2a}(-2v_1 + 2v_2) = \emptyset \quad / \cdot 2a^2$$

$$7kv_1 - 2kv_2 - 2aP_{cr}v_1 + aP_{cr}v_2 = \emptyset$$

$$kv_2 - 2kv_1 - aP_{cr}v_2 + aP_{cr}v_1 = \emptyset$$

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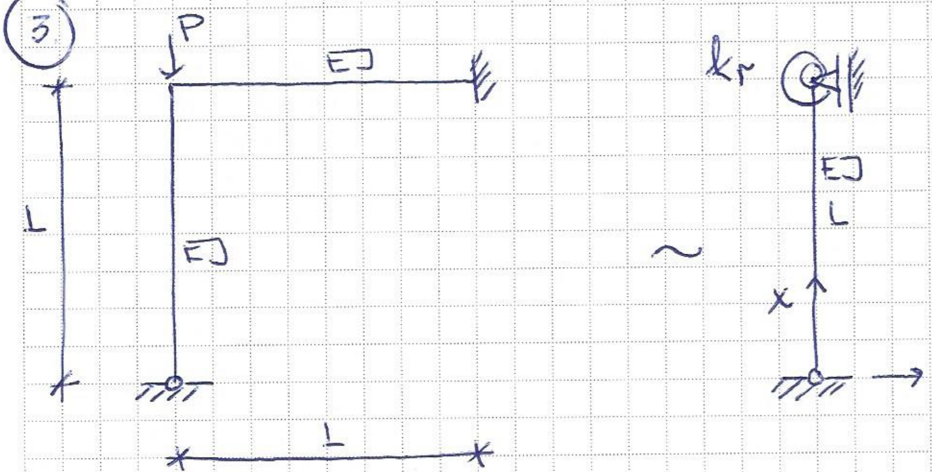
$$\begin{bmatrix} 7k - 2aP_{cr} & aP_{cr} - 2k \\ aP_{cr} - 2k & k - aP_{cr} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det[\;] = 0$$

$$7k^2 - 7akP_{cr} - 2akP_{cr} + 2a^2P_{cr}^2 - a^2P_{cr}^2 + 2akP_{cr} + 2akP_{cr} -$$
$$- 4k^2 = 3k^2 - 5akP_{cr} + a^2P_{cr}^2 = 0$$

$$P_{cr,min} = 0{,}697\,\frac{k}{a}$$

③



$$k_r = \frac{4EJ}{L}$$

(1) $y(x=0) = B + D = 0 \Rightarrow D = -B$

(2) $y''(x=0) = -\alpha^2 B = 0 \Rightarrow B = D = 0$

(3) $y(x=L) = A \sin\alpha L + CL = 0$

(4) $-EJ\,y''(x=L) = k_r \cdot y'(x=L)$

$$-EJ(-\alpha^2 A \sin\alpha L) - k_r(\alpha A \cos\alpha L + C) = 0 \quad /:EJ$$

supstitucija: $\gamma = \frac{k_r}{EJ}$

(4): $\alpha^2 A \sin\alpha L - \gamma(\alpha A \cos\alpha L + C) = 0$

$$\alpha^2 A \sin\alpha L - \alpha\gamma A \cos\alpha L - \gamma C = 0$$

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$$\begin{bmatrix} \sin\alpha L & L \\ \alpha^2 \sin\alpha L - \alpha\gamma\cos\alpha L & -\gamma \end{bmatrix} \begin{Bmatrix} A \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det[\;] = 0$$

$$-\gamma\sin\alpha L - \alpha^2 L\sin\alpha L + \alpha L\gamma\cos\alpha L = 0$$

$$\alpha\gamma L\cos\alpha L - \alpha^2 L\sin\alpha L - \gamma\sin\alpha L = 0$$

$$\gamma = \frac{k_\omega}{EJ} = \frac{\frac{4EJ}{L}}{EJ} = \frac{4}{L}$$

$$\alpha\cdot\frac{4}{L}L\cos\alpha L - \alpha^2 L\sin\alpha L - \frac{4}{L}\sin\alpha L = 0 \quad / \cdot L$$

$$4\alpha L\cos\alpha L - \alpha^2L^2\sin\alpha L - 4\sin\alpha L = 0$$

$$4\alpha L\cos\alpha L - (\alpha^2L^2+4)\sin\alpha L = 0$$

supstitucija: $u = \alpha L$:

$$f(u) = 4u\cos u - (u^2+4)\sin u = 0$$

– metodom pokušaja i pogreške:

$$u = 3{,}829 = \alpha L \Rightarrow \alpha = \frac{3{,}829}{L}$$

$$\alpha^2 = \frac{14{,}66}{L^2} = \frac{P_{cr}}{EJ} \Rightarrow P_{cr} = \frac{14{,}66\,EJ}{L^2} = \frac{1{,}485\pi^2 EJ}{L^2}$$

Za $EJ = 13600\ \text{kNm}^2$ i $L = 3{,}2\ \text{m}$

$$P_{cr} = 19.470{,}31\ \text{kN}$$

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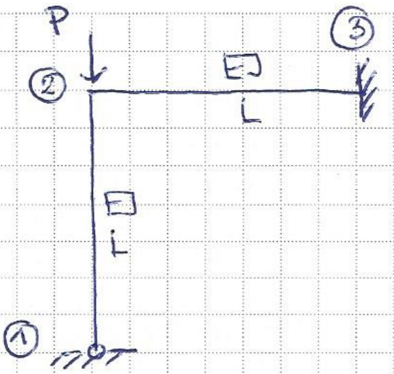
4.



Varijanta I : 2 nepoznanice : ϑ_1 i ϑ_2

$$M_{12} = r\left(\frac{EJ}{L}\right)\vartheta_1 + (rc)\left(\frac{EJ}{L}\right)\vartheta_2$$

$$M_{21} = r\left(\frac{EJ}{L}\right)\vartheta_2 + (rc)\left(\frac{EJ}{L}\right)\vartheta_1$$

$$M_{23} = 4\left(\frac{EJ}{L}\right)\vartheta_2$$

Ravnoteže čvorova :

$$M_1 = M_{12} = r\left(\frac{EJ}{L}\right)\vartheta_1 + (rc)\left(\frac{EJ}{L}\right)\vartheta_2 = \emptyset$$

$$M_2 = M_{21} + M_{23} = r\left(\frac{EJ}{L}\right)\vartheta_2 + (rc)\left(\frac{EJ}{L}\right)\vartheta_1 + 4\left(\frac{EJ}{L}\right)\vartheta_2 = \emptyset$$

$$\left(\frac{EJ}{L}\right)\begin{bmatrix} r & rc \\ rc & r+4 \end{bmatrix}\begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix} = \begin{Bmatrix} \emptyset \\ \emptyset \end{Bmatrix}$$

$$\det[\bar{K}] = r(r+4) - (rc)^2 = \emptyset$$

očitano : $\rho = 1{,}49 = \frac{P_{cr}}{P_e}$

$$P_{cr} = \frac{1{,}49\,\pi^2 EJ}{L^2}$$

za $EJ = 13600\ kNm^2$ i $L = 3{,}2m$

$$P_{cr} = 19.531{,}02\ kN$$

Varijanta II : 1 nepoznanica : ϑ_2

$$M_{21} = r'\left(\frac{EJ}{L}\right)\vartheta_2$$

$$M_{23} = 4\left(\frac{EJ}{L}\right)\vartheta_2$$

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Ravnoteža čvora 2: $M_2 = M_{21} + M_{23} = (r' + 4)\left(\frac{EJ}{L}\right)\vartheta_2 = \emptyset$

$r' = -4$

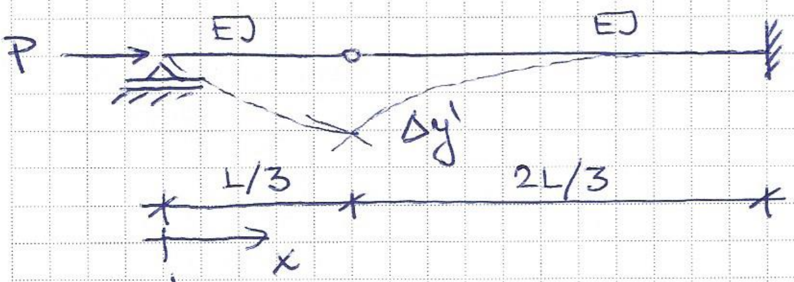
očitamo i interpoliramo: $\rho = 1,4853$

$$P_{cr} = \frac{1,4853\,\pi^2 EJ}{L^2}$$

Za $EJ = 13600\ kNm^2$ i $L = 3,2m$

$P_{cr} = 19.473,3\ kN$

(5.)



y_0'

T_0

$y_0 = M_0 = \emptyset$

(1) $M\left(x = \frac{L}{3}\right) = \alpha EJ y_0' \sin\frac{\alpha L}{3} + \frac{1}{\alpha}\sin\frac{\alpha L}{3} T_0 = \emptyset$

(2) $y(x = L) = \frac{\sin\alpha L}{\alpha} y_0' - \frac{\alpha L - \sin\alpha L}{\alpha^3 EJ} T_0 + \frac{1}{\alpha}\sin\frac{2\alpha L}{3}\Delta y' = \emptyset$

(3) $y'(x = L) = \cos\alpha L\, y_0' - \frac{1 - \cos\alpha L}{\alpha^2 EJ} T_0 + \cos\frac{2\alpha L}{3}\Delta y' = \emptyset$

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$$\begin{bmatrix} \alpha EJ \sin\frac{\alpha L}{3} & \frac{1}{\alpha}\sin\frac{\alpha L}{3} & \emptyset \\ \frac{1}{\alpha}\sin\alpha L & \frac{\sin\alpha L-\alpha L}{\alpha^3 EJ} & \frac{1}{\alpha}\sin\frac{2\alpha L}{3} \\ \cos\alpha L & \frac{\cos\alpha L-1}{\alpha^2 EJ} & \cos\frac{2\alpha L}{3} \end{bmatrix} \begin{Bmatrix} y_0' \\ T_0 \\ \Delta y' \end{Bmatrix} = \begin{Bmatrix} \emptyset \\ \emptyset \\ \emptyset \end{Bmatrix}$$

$$\det[\;] = \emptyset$$

$$\alpha EJ\sin\frac{\alpha L}{3}\left(\frac{\sin\alpha L-\alpha L}{\alpha^3 EJ}\cdot\cos\frac{2\alpha L}{3}-\frac{1}{\alpha}\sin\frac{2\alpha L}{3}\cdot\frac{\cos\alpha L-1}{\alpha^2 EJ}\right)-$$

$$-\frac{1}{\alpha}\sin\frac{\alpha L}{3}\left(\frac{1}{\alpha}\sin\alpha L\cdot\cos\frac{2\alpha L}{3}-\frac{1}{\alpha}\sin\frac{2\alpha L}{3}\cos\alpha L\right)=$$

$$=\alpha EJ\sin\frac{\alpha L}{3}\cdot\frac{\sin\alpha L}{\alpha^3 EJ}\cdot\cos\frac{2\alpha L}{3}-\alpha EJ\cdot\sin\frac{\alpha L}{3}\cdot\frac{\alpha L}{\alpha^3 EJ}\cdot\cos\frac{2\alpha L}{3}-$$

$$-\alpha EJ\sin\frac{\alpha L}{3}\cdot\frac{1}{\alpha}\sin\frac{2\alpha L}{3}\cdot\frac{\cos\alpha L}{\alpha^2 EJ}+\alpha EJ\sin\frac{\alpha L}{3}\cdot\frac{1}{\alpha}\sin\frac{2\alpha L}{3}\cdot\frac{1}{\alpha^2 EJ}-$$

$$-\frac{1}{\alpha}\sin\frac{\alpha L}{3}\cdot\frac{1}{\alpha}\sin\alpha L\cdot\cos\frac{2\alpha L}{3}+\frac{1}{\alpha}\sin\frac{\alpha L}{3}\cdot\frac{1}{\alpha}\cdot\sin\frac{2\alpha L}{3}\cdot\cos\alpha L=$$

$$=\sin\frac{\alpha L}{3}\cdot\sin\alpha L\cdot\frac{1}{\alpha^2}\cdot\cos\frac{2\alpha L}{3}-\sin\frac{\alpha L}{3}\cdot\frac{\alpha L}{\alpha^2}\cos\frac{2\alpha L}{3}-$$

$$-\sin\frac{\alpha L}{3}\cdot\frac{\sin 2\alpha L}{3}\cdot\frac{\cos\alpha L}{\alpha^2}+\sin\frac{\alpha L}{3}\cdot\sin\frac{2\alpha L}{3}\cdot\frac{1}{\alpha^2}-$$

$$-\frac{1}{\alpha^2}\sin\frac{\alpha L}{3}\cdot\sin\alpha L\cdot\cos\frac{2\alpha L}{3}+\frac{1}{\alpha^2}\sin\frac{\alpha L}{3}\cdot\sin\frac{2\alpha L}{3}\cos\alpha L=\emptyset$$

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$$\cancel{\sin\frac{\alpha L}{3}\cos\frac{2\alpha L}{3}\sin\alpha L} - \alpha L\cdot\sin\frac{\alpha L}{3}\cos\frac{2\alpha L}{3} -$$

$$-\cancel{\sin\frac{\alpha L}{3}\sin\frac{2\alpha L}{3}\cos\alpha L} + \sin\frac{\alpha L}{3}\sin\frac{2\alpha L}{3} -$$

$$-\cancel{\sin\frac{\alpha L}{3}\cos\frac{2\alpha L}{3}\sin\alpha L} + \cancel{\sin\frac{\alpha L}{3}\sin\frac{2\alpha L}{3}\cos\alpha L} =$$

$$=\sin\frac{\alpha L}{3}\sin\frac{2\alpha L}{3} - \alpha L\sin\frac{\alpha L}{3}\cos\frac{2\alpha L}{3} =$$

$$=\sin\frac{\alpha L}{3}\left(\sin\frac{2\alpha L}{3} - \alpha L\cdot\cos\frac{2\alpha L}{3}\right) = \emptyset$$

$$u = \alpha L$$

$$f(u) = \sin\frac{u}{3}\left(\sin\frac{2u}{3} - u\cos\frac{2u}{3}\right) = 0$$ ili samo $f(u) = \sin\frac{2u}{3} - u\cos\frac{2u}{3}$

$\rightarrow$ metodom pokušaja i pogreške:

$$u = 1{,}45$$

$$\alpha = \frac{1{,}45}{L} \Rightarrow \alpha^2 = \frac{2{,}10}{L^2} = \frac{P_{cr}}{EJ}$$

$$P_{cr} = \frac{2{,}10\,EJ}{L^2} = \frac{0{,}213\,\pi^2 EJ}{L^2}$$