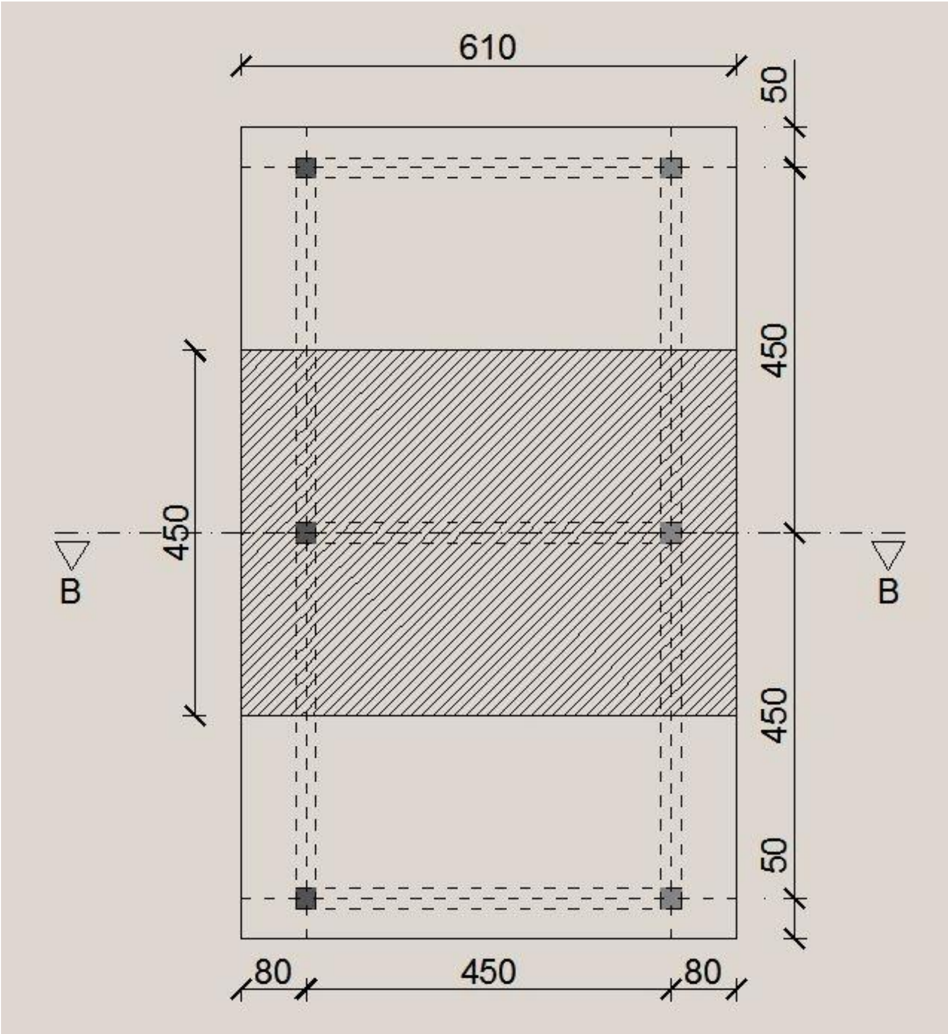


NAPOMENA:

Sve matrice su množene pomoću programa „Matrix Calc", kubna jednadžba riješena preko web-a, determinanta riješena pomoću programa „Wolfram", tri jednađbe s tri nepoznanice su riješene pomoću programa „Gaus", a matrica popustljivosti je riješena u programu „Robot Structural Analysis".

Početni podaci:

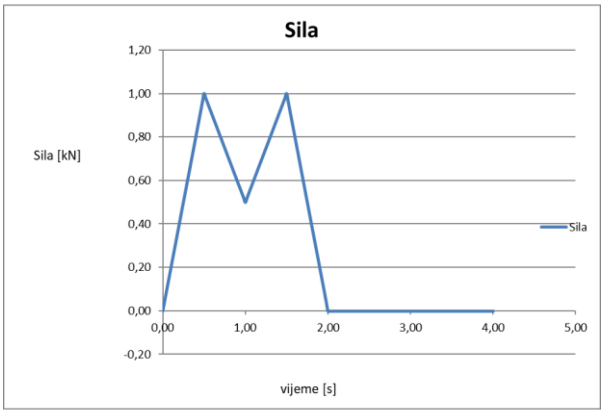
Materijal:
Beton klase C 30/37

Opterećenje na nivou međukatnih konstrukcija:

- Vlastita težina konstrukcija
- Stalno: g = 4,00 kN/m^2
- Promjenjivo: p = 5,00 kN/m^2, (γ_p = 0,70)

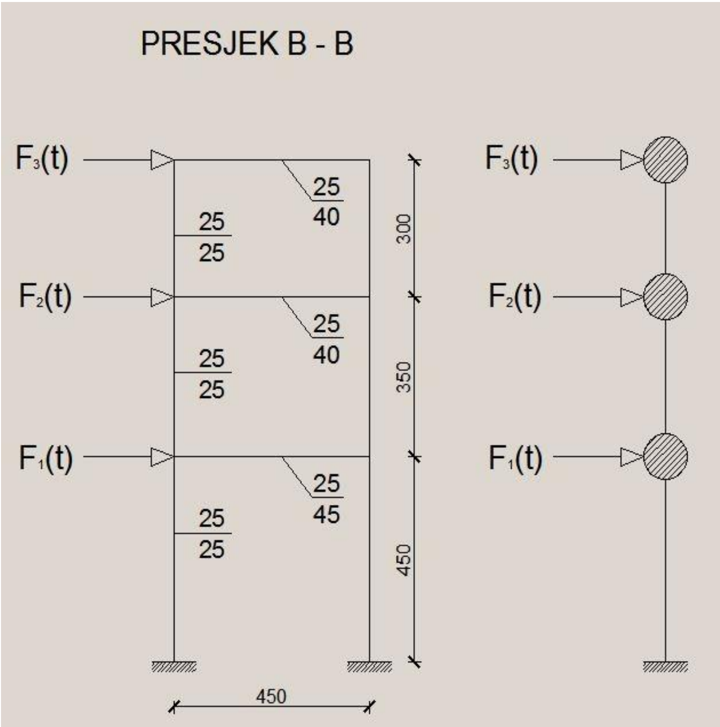
Prigušenje: 1,50%
Amplituda sile: F_0 = 30 kN

$$F = \begin{Bmatrix} 1,0 \\ 1,2 \\ 1,3 \end{Bmatrix} F_0$$



Debljina armiranobetonske ploče: d=22cm

Pri proračunu masa, promjenjivo opterećenje uzeti u iznosu od 70%



A) Početni (pripremni) proračun

1. Matrica masa:

$$m = \frac{(g + \gamma p + d \times \gamma_c) \times A_i}{a_g} = \frac{(4,00 + 0,70 \times 5,00 + 0,22 \times 25) \times (4,5 \times 6,1)}{9,81} = 36,38t$$

$$m_1 = [0,25 \times 0,45 \times 4,5 + 0,25 \times 0,25 \times 6,25 \times 2] \times 2,5 = 3,22 \quad t$$

$$m_2 = [0,25 \times 0,40 \times 4,5 + 0,25 \times 0,25 \times 3,25 \times 2] \times 2,5 = 2,14 \quad t$$

$$m_3 = [0,25 \times 0,40 \times 4,5 + 0,25 \times 0,25 \times 1,5 \times 2] \times 2,5 = 1,59 \quad t$$

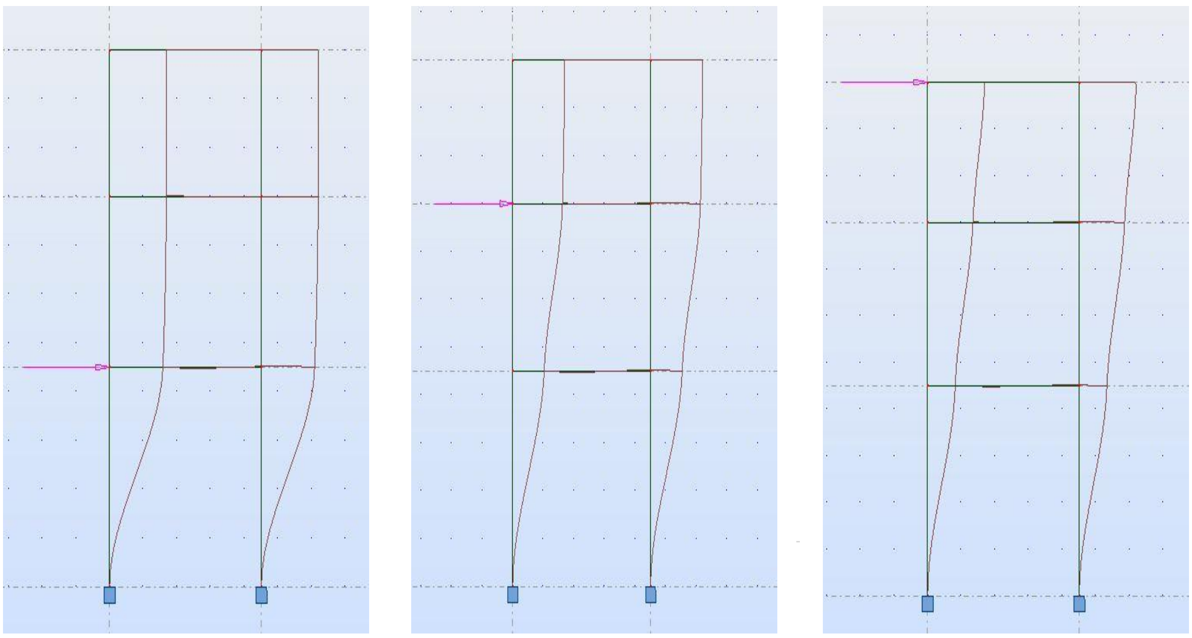
$$m_A = 36,38 + 3,22 = 39,60 \quad t$$

$$m_B = 36,38 + 2,14 = 38,52 \quad t$$

$$m_C = 36,38 + 1,59 = 37,97 \quad t$$

$$[m] = \begin{bmatrix} 39,60 & 0 & 0 \\ 0 & 38,52 & 0 \\ 0 & 0 & 37,97 \end{bmatrix}$$

2. Matrica popustljivosti (iz Robota očitano):



3. Vlastite kružne frekvencije (ω), vlastiti periodi (T) i frekvencije (f), Vlastiti (prirodni) oblici

Dinamička matrica:

$$[D]=[a]\times[m]=\begin{bmatrix}0,00038268 & 0,00040784 & 0,00041025\\ 0,00040783 & 0,00064415 & 0,00067207\\ 0,00041023 & 0,00067206 & 0,00084516\end{bmatrix}\times\begin{bmatrix}39,60 & 0 & 0\\ 0 & 38,52 & 0\\ 0 & 0 & 37,97\end{bmatrix}$$

$$[D]=\begin{bmatrix}0,015154 & 0,015710 & 0,015577\\ 0,016150 & 0,024813 & 0,025518\\ 0,016245 & 0,025888 & 0,032091\end{bmatrix}$$

$$\det\left[[D]-\lambda[I]\right]=0$$

$$\det\begin{bmatrix}(d_{11}-\lambda) & d_{12} & d_{13}\\ d_{21} & (d_{22}-\lambda) & d_{23}\\ d_{31} & d_{32} & (d_{33}-\lambda)\end{bmatrix}=0$$

$$\det\begin{bmatrix}(0,015154\text{-}\lambda) & 0,015710 & 0,015577\\ 0,016150 & (0,024813-\lambda) & 0,025518\\ 0,016245 & 0,025888 & (0,032091-\lambda)\end{bmatrix}=0$$

$$\lambda^3-0,072058\lambda^2+0,000491222\lambda-6,59976\times10^{-7}=0$$

$$\lambda_1=0,064613629\quad\Rightarrow\omega_1=\sqrt{\frac{1}{\lambda_1}}=3,934rad/s\quad\Rightarrow T_1=\frac{2\pi}{\omega_1}=1,597s\quad\Rightarrow f_1=\frac{1}{T_1}=0,626Hz$$

$$\lambda_2=0,005630187\quad\Rightarrow\omega_2=\sqrt{\frac{1}{\lambda_2}}=13,327rad/s\quad\Rightarrow T_2=\frac{2\pi}{\omega_2}=0,471s\quad\Rightarrow f_2=\frac{1}{T_2}=2,123Hz$$

$$\lambda_3=0,001814183\quad\Rightarrow\omega_3=\sqrt{\frac{1}{\lambda_3}}=23,478rad/s\quad\Rightarrow T_3=\frac{2\pi}{\omega_3}=0,267s\quad\Rightarrow f_3=\frac{1}{T_3}=3,745Hz$$

Karakteristični vektori oblika:

$$\left([D]-\lambda[I]\right)\times\{u\}=\{0\}$$

a) I. *vektor oblika {u}₁ za λ=λ₁*

$$(d_{11}-\lambda_1)u_{11}+d_{12}u_{21}+d_{13}u_{31}=0$$
$$d_{21}u_{11}+(d_{22}-\lambda_1)u_{21}+d_{23}u_{31}=0$$
$$d_{31}u_{11}+d_{32}u_{21}+(d_{33}-\lambda_1)u_{31}=0$$

$$(0,015154-0,064613629)\times u_{11}+0,015710\times u_{21}+0,015577\times u_{31}=0 \quad (1)$$

$$0,016150\times u_{11}+(0,024813-0,064613629)\times u_{21}+0,025518\times u_{31}=0 \quad (2)$$

$$0,016245\times u_{11}+0,025888\times u_{21}+(0,032091-0,064613629)\times u_{31}=0 \quad (3)$$

Za pretpostavljeni u_{31} = 1,0 iz jednadžbi 2 i 3 dobije se:

$$\{u\}_1=\begin{Bmatrix}0,595\\0,883\\1,0\end{Bmatrix}$$

b) *vektor oblika {u}₂ za λ=λ₂*

$$(d_{11}-\lambda_2)u_{12}+d_{12}u_{22}+d_{13}u_{32}=0$$
$$d_{21}u_{12}+(d_{22}-\lambda_2)u_{22}+d_{23}u_{32}=0$$
$$d_{31}u_{12}+d_{32}u_{22}+(d_{33}-\lambda_2)u_{32}=0$$

$$(0,015154-0,005630187)\times u_{12}+0,015710\times u_{22}+0,015577\times u_{32}=0 \quad (1)$$

$$0,016150\times u_{12}+(0,024813-0,005630187)\times u_{22}+0,025518\times u_{32}=0 \quad (2)$$

$$0,016245\times u_{12}+0,025888\times u_{22}+(0,032091-0,005630187)\times u_{32}=0 \quad (3)$$

Za pretpostavljeni u_{32} = 1,0 iz jednadžbi 2 i 3 dobije se:

$$\{u\}_2 = \begin{Bmatrix} -1,437 \\ -0,120 \\ 1,0 \end{Bmatrix}$$

c) ***vektor oblika {u}₃ za λ=λ₃***

$$\left(d_{11} - \lambda_3\right)u_{13} + d_{12}u_{23} + d_{13}u_{33} = 0$$
$$d_{21}u_{13} + \left(d_{22} - \lambda_3\right)u_{23} + d_{23}u_{33} = 0$$
$$d_{31}u_{13} + d_{32}u_{23} + \left(d_{33} - \lambda_3\right)u_{33} = 0$$

$$\left(0,015154 - 0,001814183\right) \times u_{12} + 0,015710 \times u_{22} + 0,015577 \times u_{32} = 0 \quad (1)$$

$$0,016150 \times u_{12} + \left(0,024813 - 0,001814183\right) \times u_{22} + 0,025518 \times u_{32} = 0 \quad (2)$$

$$0,016245 \times u_{12} + 0,025888 \times u_{22} + \left(0,032091 - 0,001814183\right) \times u_{32} = 0 \quad (3)$$

Za pretpostavljeni u_{33} = 1,0 iz jednadžbi 2 i 3 dobije se:

$$\{u\}_3 = \begin{Bmatrix} 0,803 \\ -1,674 \\ 1,0 \end{Bmatrix}$$

Ortonormiranje vektora oblika:

$$\{\bar{u}\}_i^T = [m]\{\bar{u}\}_j = \begin{cases} 1 & i = 1 \\ 0 & i \neq j \end{cases}$$

$$\{\bar{u}\}_i = \alpha_i \{u\}_i$$

$$\alpha_i = \sqrt{\frac{1}{\{u\}_i^T [m]\{u\}_i}}$$

$$\{u\}_1 = \{0{,}595 \quad 0{,}883 \quad 1{,}0\}\, x \begin{bmatrix} 39{,}60 & 0 & 0 \\ 0 & 38{,}52 & 0 \\ 0 & 0 & 37{,}97 \end{bmatrix} x \begin{Bmatrix} 0{,}595 \\ 0{,}883 \\ 1{,}0 \end{Bmatrix}$$

$$= 82{,}0229$$

$$\alpha_1 = 0{,}11042$$

$$\{u\}_2 = \{-1{,}437 \quad -0{,}120 \quad 1{,}0\}\, x \begin{bmatrix} 39{,}60 & 0 & 0 \\ 0 & 38{,}52 & 0 \\ 0 & 0 & 37{,}97 \end{bmatrix} x \begin{Bmatrix} -1{,}437 \\ -0{,}120 \\ 1{,}0 \end{Bmatrix}$$

$$= 120{,}337$$

$$\alpha_2 = 0{,}09116$$

$$\{u\}_3 = \{0{,}803 \quad -1{,}674 \quad 1{,}0\}\, x \begin{bmatrix} 39{,}60 & 0 & 0 \\ 0 & 38{,}52 & 0 \\ 0 & 0 & 37{,}97 \end{bmatrix} x \begin{Bmatrix} 0{,}803 \\ -1{,}674 \\ 1{,}0 \end{Bmatrix}$$

$$= 171{,}436$$

$$\alpha_3 = 0{,}07637$$

Te su ortonomirani vektori oblika, redom:

$$\{\bar{u}\}_1 = \begin{Bmatrix} 0{,}0657353 \\ 0{,}0974685 \\ 0{,}1104183 \end{Bmatrix} \{\bar{u}\}_2 = \begin{Bmatrix} -0{,}131026 \\ -0{,}010955 \\ 0{,}0911590 \end{Bmatrix} \{\bar{u}\}_3 = \begin{Bmatrix} 0{,}061357 \\ -0{,}12783 \\ 0{,}076374 \end{Bmatrix}$$

4. Modalna matrica

$$[\Phi] = \begin{bmatrix} 0{,}065735 & -0{,}131026 & 0{,}061357 \\ 0{,}097468 & -0{,}010955 & -0{,}127829 \\ 0{,}110418 & 0{,}091159 & 0{,}0763745 \end{bmatrix}$$

5. Odabir vremenskog koraka integracije

$T_n = T_1 = 1,597$

$$\Delta t \leq \frac{T_n}{10} = \frac{1,597}{10} = 0,1597 s$$

Odabrano: Δt = 0,02 s

B) Iterativni postupak

1. Transformacija sustava u sustav nezavisnih diferencijalnih jednažbi tzv. Modalnih jednadžbi

$$[\mathrm{m}]\{\ddot{\mathrm{x}}\}+[\mathrm{c}]\{\dot{\mathrm{x}}\}+[\mathrm{k}]\{\mathrm{x}\}=\{\mathrm{F}(\mathrm{t})\} \qquad / \ [\Phi]^{\mathrm{T}} \qquad / \ [\Phi][\Phi]^{-1}$$

$$[\Phi]^{\mathrm{T}}[\mathrm{m}][\Phi][\Phi]^{-1}\{\ddot{\mathrm{x}}\}+[\Phi]^{\mathrm{T}}[\mathrm{c}][\Phi][\Phi]^{-1}\{\dot{\mathrm{x}}\}+[\Phi]^{\mathrm{T}}[\mathrm{k}][\Phi][\Phi]^{-1}\{\mathrm{x}\}=[\Phi]^{\mathrm{T}}\{\mathrm{F}(\mathrm{t})\}$$

$$[\mathrm{I}]\{\ddot{\eta}\}+[2\xi\omega]\{\dot{\eta}\}+[\omega^2]\{\eta\}=\mathrm{f}(\mathrm{t})$$

$$\{f(t)\}=[\Phi]^T\times\{F(t)\}$$

$$\{F(t)\}=F_0\times\begin{Bmatrix}1,0\\1,2\\1,3\end{Bmatrix}$$

$$F_0=30kN$$

$$\{F(t)\}=\begin{Bmatrix}30\\36\\39\end{Bmatrix}$$

$$\{f(t)\}=\begin{bmatrix}0,065735 & 0,097469 & 0,110418\\ -0,13103 & -0,01096 & 0,091159\\ 0,061357 & -0,12783 & 0,076375\end{bmatrix}\times\begin{Bmatrix}30\\36\\39\end{Bmatrix}=\begin{Bmatrix}9,787245\\-0,76999\\0,217457\end{Bmatrix}$$

2. Modalne diferencijalne jednadžbe

$$\ddot{\eta}_1+2\xi\omega_1\dot{\eta}_1+\omega_1^2\eta_1=f_1(t)$$

$$\ddot{\eta}_2+2\xi\omega_2\dot{\eta}_2+\omega_2^2\eta_2=f_2(t)$$

$$\ddot{\eta}_3+2\xi\omega_3\dot{\eta}_3+\omega_3^2\eta_3=f_3(t)$$

$\Delta t = 0,02s$
$\zeta = 0,015$
$\omega_1 = 3,934 rad/s$
$\omega_2 = 13,327 rad/s$
$\omega_3 = 23,478 rad/s$
$n = \zeta \times \omega$

Pomak u trenutku t_{i+1}:

$$x_{i+1} = e^{-\eta\Delta t} \cdot \left[x_i \cdot \cos(\omega_D \Delta t) + \frac{\dot{x}_i + n x_i}{\omega_D} \cdot \sin(\omega_D \Delta t) \right] + \frac{F_i}{k}\left[1 - e^{-\eta\Delta t}\left(\cos(\omega_D \Delta t) + \frac{n}{\varpi}\sin(\omega_D \Delta t) \right) \right] +$$
$$+ \frac{F_{i+1} - F_i}{k\Delta t}\left[\Delta t - \frac{2n}{\omega_D^2} + e^{-\eta\Delta t}\left(2\frac{n}{\omega^2}\cos(\omega_2 \Delta t) - \frac{\omega^2 - n^2}{\omega_D \omega} \cdot \sin(\omega_D \Delta t) \right) \right]$$

Brzina u trenutku t_{i+1}:

$$v_{i+1} = e^{-\eta\Delta t} \cdot \left[\dot{x}_i \cdot \cos(\omega_D \Delta t) - \frac{n\dot{x}_i + \omega^2 x_i}{\omega_D} \cdot \sin(\omega_D \Delta t) \right] + \frac{F_i}{k} \cdot \frac{\omega^2}{\omega_D} \cdot e^{-\eta\Delta t} \cdot \sin(\omega_D \Delta t) +$$
$$+ \frac{F_{i+1} - F_i}{k\Delta t}\left[1 - e^{-\eta\Delta t}\left(\cos(\omega_D \Delta t) + \frac{n}{\omega_D} \cdot \sin(\omega_D \Delta t) \right) \right]$$

Prethodne jednadžbe se mogu zapisati u obliku:

$$x_{i+1} = A \times x_i + B \times \dot{x}_i + C \times p_i + D \times p_{i+1}$$
$$\dot{x}_{i+1} = A' \times x_i + B' \times \dot{x}_i + C \times p_i + D \times p_{i+1}$$

gdje su:

$$A = e^{-\xi\omega\Delta t}\left(\frac{1}{\sqrt{1-\xi^2}} \sin \omega_D \Delta t + \cos \omega_D \Delta t \right)$$

$$B = e^{-\xi\omega\Delta t}\left(\frac{1}{\omega_D} \sin \omega_D \Delta t \right)$$

$$C = \frac{1}{k}\left\{ \frac{2\xi}{\omega\Delta t} + e^{-\xi\omega\Delta t}\left[\left(\frac{1-2\xi^2}{\omega_D \Delta t} - \frac{\xi}{\sqrt{1-\xi^2}} \right) \sin \omega_D \Delta t - \left(1 + \frac{2\xi}{\omega\Delta t} \right) \cos \omega_D \Delta t \right] \right\}$$

$$D = \frac{1}{k}\left[1 - \frac{2\xi}{\omega\Delta t} + e^{-\xi\omega\Delta t}\left(\frac{2\xi - 1}{\omega_D \Delta t} \sin \omega_D \Delta t + \frac{2\xi}{\omega\Delta t} \right) \cos \omega_D \Delta t \right]$$

$$A' = -e^{-\xi\omega\Delta t}\left(\frac{\omega}{\sqrt{1-\xi^2}}\sin\omega_D\Delta t\right)$$

$$B' = e^{-\xi\omega\Delta t}\left(\cos\omega_D\Delta t - \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D\Delta t\right)$$

$$C' = \frac{1}{k}\left\{-\frac{1}{\Delta t} + e^{-\xi\omega\Delta t}\left[\left(\frac{\omega}{\sqrt{1-\xi^2}} + \frac{\xi}{\Delta t\sqrt{1-\xi^2}}\right)\sin\omega_D\Delta t + \frac{1}{\Delta t}\cos\omega_D\Delta t\right]\right\}$$

$$D' = \frac{1}{k}\left[1 - e^{-\xi\omega\Delta t}\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D\Delta t + \cos\omega_D\Delta t\right)\right]$$

3. Određivanje stvarnih pomaka i brzina

Inverznom transformacijom modalnih pomaka, brzina i ubrzanja dobiju se stvarni generalizirani pomaci, brzine i ubrzanja.

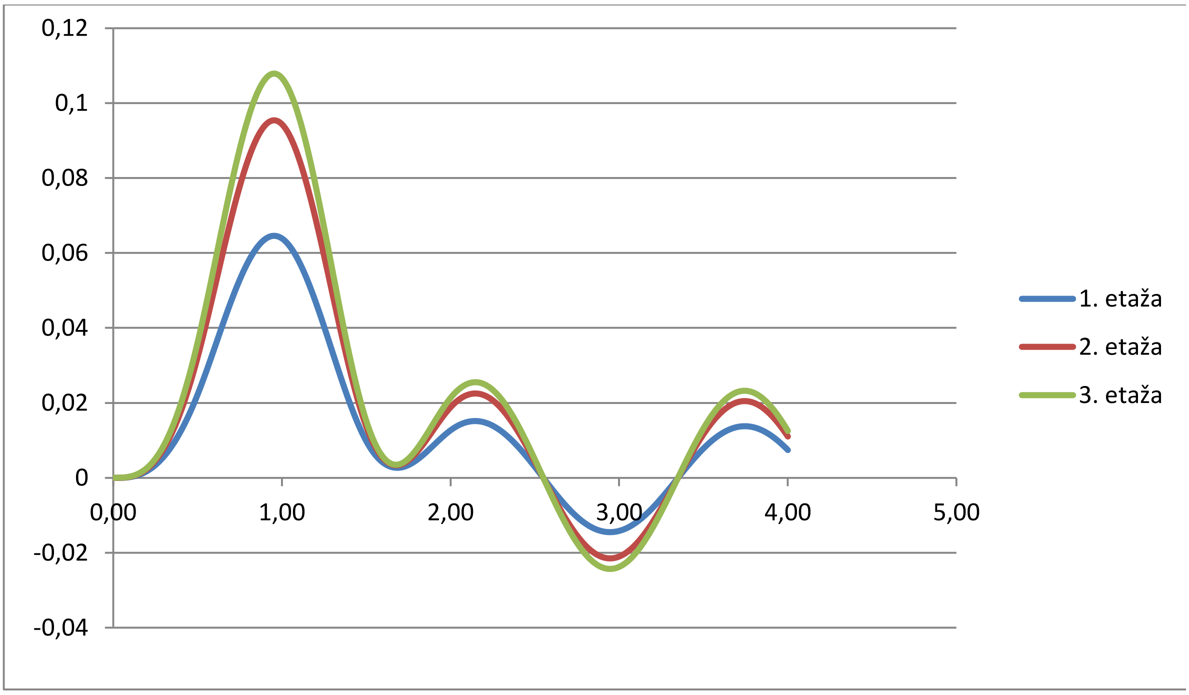
$$[x] = [\phi]\times[\eta]_i$$

$$[x] = \begin{bmatrix} 0,064327 & -0,131017 & 0,061325 \\ 0,095463 & -0,010940 & -0,127845 \\ 0,108113 & 0,091174 & 0,076371 \end{bmatrix} \times \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix}$$

$$[x_I] = 0,064327\times\eta_1 - 0,131017\times\eta_2 + 0,061325\times\eta_3$$

$$[x_{II}] = 0,095463\times\eta_1 - 0,010940\times\eta_2 - 0,127845\times\eta_3$$

$$[x_{III}] = 0,108113\times\eta_1 + 0,091174\times\eta_2 + 0,076371\times\eta_3$$



NAPOMENA:

Svaka jednadžba riješena je zasebno u programskom paketu „Excel“ metodom interpolacije uzbudne sile, a stvarni pomaci su dobiveni inverznom transformacijom modalnih pomaka pomoću modalne matrice.